Problem Set 8

Released: April 10, 2023
Due: April 21, 2023, 11:59pm

Submit your solutions to this homework on Canvas in a group of 2 or 3. Your solutions must be submitted in pdf produced using LaTeX.

Definition 1. *A judgmental model of CBPV*[1] *consists of*

1. *A "value category": cartesian category $\mathscr{V}$, whose objects we typically write as A*
2. *A "computation category": category $\mathscr{E}$ whose objects we typically write as B*
3. *An "empty computation object": object $I \in \mathscr{E}$*
4. *"function objects": a functor $\to : \mathscr{V}^o \times \mathscr{E} \to \mathscr{E}$ with natural isomorphisms*[2]

$$(A_1 \times A_2) \to B \cong A_1 \to A_2 \to B$$

   and

$$1 \to B \cong B$$

Given a judgmental CBPV model, we can define various objects by universal properties.

1. *For a $A \in \mathscr{V}$, a* return object *for A is an object $RetA \in \mathscr{E}$ with a natural isomorphism*
$$\mathscr{E}(RetA, B) \cong \mathscr{E}(I, A \to B)$$
2. *For a $B \in \mathscr{E}$, a* closure object *for B is an object $CloB \in \mathscr{V}$ with a natural isomorphism*
$$\mathscr{V}(A, CloB) \cong \mathscr{E}(I, A \to B)$$
3. *For $A \in \mathscr{V}$ and $B \in \mathscr{E}$, a* tensor object *for A and B, is an object $A \oslash B \in \mathscr{E}$ with a natural isomorphism*
$$\mathscr{E}(A \oslash B, B') \cong \mathscr{E}(B, A \to B')$$

Notice that if all return and closure objects exist that they are adjoint $Ret \dashv Clo$

[1] note that these models are not the most general, there is a weaker notion based on CT structures we will discuss in class

[2] In the full definition these natural isomorphisms should be subject so some equations but we will elide those details here

Problem 1 Models for Programs with Errors

Let $\mathscr{C}$ be a bicartesian closed category and let $E \in \mathscr{C}$. Define the *E-error model* as follows:

1. The value category is $\mathscr{C}$
2. The computation category is $E/\mathscr{C}$ (see Riehl exercise 1.1.iii)
3. The empty computation object is the left injection $\text{in}_0 : E \to E + 1$
4. The function object $A \to (B, e : \mathscr{C}(E, B))$ is defined as

$$B^A, \lambda(e \circ \pi_1^{E,A})$$

Show that the E-error model has all return objects and closure objects. (HINT: use the Yoneda lemma to simplify the definition of return and closure objects)

. . . .

Problem 2 Models for Programs with Continuations

Let $\mathscr{C}$ be a cartesian closed category and let $R \in \mathscr{C}$. Define the *R-continuation model* as follows:

1. The value category is $\mathscr{C}$
2. The computation category is $\mathscr{C}^{op}$
3. The empty computation object is R
4. The function object $A \to B$ is $A \times B$

Show that the R-continuation model has all return objects and closure objects. (HINT: use the Yoneda lemma to simplify the definition of return and closure objects)

. . . .

Problem 3 Models for Programs with State (and other effects)

Let $(\mathscr{V}, \mathscr{E}, I, \to)$ be a judgmental CBPV model, and let $S \in \mathscr{V}$ be an object and $S \oslash I$ be a tensor of S and I.

Define the *S-stateful model* as follows:

1. The value category is the same, $\mathscr{V}$
2. The computation category is the same, $\mathscr{C}$

3. The empty computation object is $S \oslash I$
4. The function object $A \to B$ is the same, $A \to B$

1. Show that if the original model has all return objects then so does the S-stateful model.
2. Show that if the original model has all closure objects then so does the S-stateful model.

(HINT: use the Yoneda lemma to simplify the definition of return and closure objects)

....