Problem Set 1: Logic and Order Theory

Released: January 9, 2023
Due: January 23, 2022, 11:59pm

Submit your solutions to this homework on Canvas in a group of 2 or 3. Your solutions must be submitted in pdf produced using LaTeX.

If you haven't already, sign up to scribe and present homework solutions on the course gitlab repo.

Problem 1 Distributivity

A lattice (poset with finite meets and joins) is *distributive* when binary meets and joins satisfy a distributive law:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

We say that propositions P and Q of a logic are *equivalent* when the judgments $P \vdash Q$ and $Q \vdash P$ are both provable.

1. Show that IPL satisfies this distributive law in that for any propositions A, B, C, the propositions $A \wedge (B \vee C)$ and $(A \wedge B) \vee (A \wedge C)$ are equivalent.

2. Show that any Heyting lattice (poset with finite meets, joins and an implication operation) is a distributive lattice.

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Problem 2 Intuitionistic and Classical Logic

The law of excluded middle or principle of omniscience is the following axiom scheme: for all propositions A the axiom

$$\frac{}{\Gamma \vdash A \vee \neg A}$$

The law of double negation elimination is the axiom scheme

$$\frac{}{\Gamma \vdash \neg(\neg A) \supset A}$$

Since the first model of intuitionistic logic that is not a boolean algebra is a 3 element Heyting algebra, it is easy to get the impression that intuitionistic logic is about "multi-valued logics" where there is some "third" truth value other than just true and false. But this is not entirely accurate: for instance there are many boolean algebras with more than 2 elements (the powerset of any set) that are useful models of classical logic. Furthermore, while there might be more than 2 elements in a model, *within* the logic, we can never separate any proposition from true and false.

1. Show that in IPL extended with the law of excluded middle, the law of double negation elimination is admissible and vice-versa.

2. The following might be called the "intuitionistic law of excluded middle", for all Γ, A:

$$\frac{}{\Gamma \vdash \neg(\neg(A \Longleftrightarrow \top) \land \neg(A \Longleftrightarrow \bot))}$$

   Intuitively this says "no proposition is not equivalent to true and not equivalent to false", where we are using the notations $\neg B = (B \supset \bot)$ and $B \Longleftrightarrow C = (B \supset C) \land (C \supset B)$.

   Show that the intuitionistic law of excluded middle is derivable for all A in IPL. A full proof tree for this will be quite large, so I encourage you to develop intermediate reasoning principles to make this proof clearer.

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Problem 3 Conservativity Results

Fix a set of propositional variables, but no axioms. In this problem we will prove that IPL($\top, \land, \supset$) is a *conservative extension* of IPL($\top, \land$)[1]. That is, we will show that for any judgment $\Gamma \vdash A$ where the propositions in Γ, A are generated using only $\top, \land$ and propositional variables, if there is a proof that uses the implication $\supset$ then there is a proof that doesn't use it. In other words, $\supset$ doesn't let us prove anything new about propositions that don't involve $\supset$. This means that we can use the richer logic of IPL($\top, \land, \supset$) to prove results that hold in any poset with finite meets, even those that don't support an implication structure. And while $\supset$ can't allow us to prove anything new, it might allow us to write a *shorter* or more *intuitive* proof.

To prove conservativity, we will show that the inclusion (a monotone function of posets) i : IPL($\top, \land$) $\to$ IPL($\top, \land, \supset$) is an *order embedding*: if $i(P) \leq i(Q)$ then $P \leq Q$. Recall that the ordering here is provability of the hypothetical judgment $\vdash$, so this means if $P \vdash Q$ in IPL($\top, \land, \supset$) then $P \vdash Q$ in IPL($\top, \land$).

Key to this proof is the *initiality* property of each variant of IPL:

[1]This conservativity result also holds when both have disjunction, but the proof is slightly more complex

- For any poset P with finite meets and an assignment $\sigma(X) \in P$ for each propositional variable, there is a unique monotone function $\overline{\sigma} : \mathrm{IPL}(\top, \wedge) \to P$ that preserves finite meets and respects the assignment of propositional variables $\overline{\sigma}(X) = \sigma(X)$.

- An analogous property holds for $\mathrm{IPL}(\top, \wedge, \supset)$ but the poset P must have an implication as well and $\overline{\sigma}$ is the unique monotone function preserving finite meets, respecting the assignment σ and preserving the implication.

1. First show that for any poset P, the set of contravariant homomorphisms Bool^{P^o}, with the point-wise ordering:

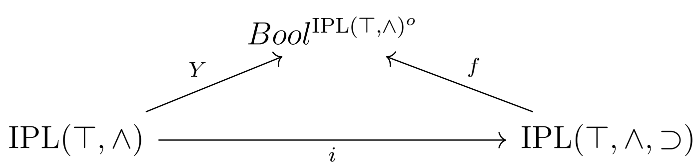
$$f \leq g = \forall x \in P. f(x) \leq g(x)$$

   has finite meets and an implication.

2. Next, for any P we can define a function $Y : P \to \mathrm{Bool}^{P^o}$ defined by $Y(x)(y) = y \leq x$. Show that
   - Y is monotone.
   - Y preserves any finite meets that exist.
   - Y is an order embedding: if $Y(x) \leq Y(y)$ then $x \leq y$.

3. Show that if $i : P \to Q$ and $j : Q \to R$ are monotone functions and $j \circ i : P \to R$ is an order embedding then i is an order embedding.

4. Use the initiality property of $\mathrm{IPL}(\top, \wedge, \supset)$ to construct a monotone function $f : \mathrm{IPL}(\top, \wedge, \supset) \to Bool^{\mathrm{IPL}(\top, \wedge)^o}$ that makes the following diagram commute:



   Note that you will need to use the initiality of $IPL(\top, \wedge)$ to show the diagram commutes.

Then since Y is an embedding, we have that the inclusion i is an embedding.

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