Lecture 14: C-T Structures III, Soundness and Completeness

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Today's Lecture is to finally wrap up the semantics of simple type theory. We've already covered the semantics of product and function types, and thus we can finalize them by discussing the sum types and the empty type. The universal properties we've been talking about products and functions, they correspond to products, exponential and category, whereas when we talk about co-products, these are dual of products. So they are in terms of what we could call contra variant predicators on the opposite category. It turns out that for this reason, explaining the semantics of the co-products is more complicated. So, in order to do it in as efficient way of possible, we first introduce an auxiliary definition. We'll see how to define co-products from there.

1 Category of Unary Terms

Let S be C-T Structures (notice that S has a notion of context and a notion of term). Fix an object $\Gamma \in S_c$ in the context category. After defining the category Un_Γ, the idea is as follows, the objects of the unary category are going to be all of the types. We are defining different category for each Γ, but the category will have the same set of objects for each Γ. The objects are the types of C-T structure and the morphisms from A to B are going to be terms in S of B. But in the context, Γ times sole of A, where this sole is our context in S that represents the term of type A.

Definition 1 (Category of Unary Terms)**.** *Fix* $\Gamma \in S_c$*. Define a category* Un_Γ *as follows*

- $(\mathrm{Un}_\Gamma)_0 = S_T$*, i.e., the objects are all types*
- $\mathrm{Un}_\Gamma(A, B) = \mathrm{Tm}B(\Gamma \times \mathrm{sole}A)$*, i.e., the terms are terms of type* B *with "inputs" from* Γ *and* A*.*
- $\mathrm{id}_A = \mathrm{var} * \pi_2^{\Gamma,\mathrm{sole}A}$
- $M \circ N = M * (\pi_1, N/\mathrm{var})$

- *Left unit:*

$$\begin{aligned}(\mathrm{var} * \pi_2) * (\pi_1, N/\mathrm{var}) &= \mathrm{var} * (\pi_2 \circ (\pi_1, N/\mathrm{var}) && \text{(predicator associativity)}\\ &= \mathrm{var} * N/\mathrm{var} && \text{(property of products)}\\ &= N && \text{(property of sole)}\end{aligned}$$

- *Right unit:*

$$\begin{aligned}M * (\pi_1, (\mathrm{var} * \pi_2)/\mathrm{var}) &= M * (\pi_1, \pi_2) && \text{(property of sole)}\\ &= M * \mathrm{id} && \text{(property of products)}\\ &= M && \text{(predicator unit)}\end{aligned}$$

- *Associativity:*

$$\begin{aligned}M * (\pi_1, (N * (\pi_1, P/\mathrm{var}))/\mathrm{var}) &= M * (\pi_1, N/\mathrm{var} \circ (\pi_1, P/\mathrm{var})))\\ &\qquad \text{(Naturality of } -/\mathrm{var})\\ &= M * (\pi_1 \circ (\pi_1, P/\mathrm{var}), N/\mathrm{var} \circ (\pi_1, P/\mathrm{var}))\\ &\qquad \text{(property of products)}\\ &= M * ((\pi_1, N/\mathrm{var}) \circ (\pi_1, P/\mathrm{var}))\\ &\qquad \text{(naturality of } (-,=))\\ &= (M * (\pi_1, N/\mathrm{var})) * (\pi_1, P/\mathrm{var})\\ &\qquad \text{(predicator associativity)}\end{aligned}$$

Thinking about this in terms of the syntactic CT structure $\mathscr{L}$, we fix a real context Γ. Then the objects Un_Γ are just types in STT and the morphisms from A to B is a term M with a free variable for x: $\Gamma, x : A \vdash M : B$. And the idea for M is that we can define composition of such terms by substitution along this single variable, so we can get that:

$$\frac{\Gamma, x : B \vdash M : C \quad \Gamma, x : A \vdash N : B}{\Gamma, x : A \vdash M[N/x] : C}$$

And the identity is just the variable:

$$\Gamma, x : A \vdash x : A$$

And the unary category generalizes this construction to an arbitrary C-T Structure.

1.1 Functors derived from construction

The first thing we should notice from this construction is that it's parameterized by Γ. So it's reasonable to ask can we get some functors between these categories, if we have a substitution, like morphisms between these Γ. We do, but note that this is contravariant:

Definition 2. *Given $\gamma \in S_c(\Delta, \Gamma)$, we define a functor*

$$\mathrm{Un}_\gamma : \mathrm{Un}_\Gamma \to \mathrm{Un}_\Delta$$

as follows:

- $\mathrm{Un}_\gamma(A) = A$
- $\mathrm{Un}_\gamma(M) = M * (\gamma \circ \pi_1, \pi_2)$
- *Preserves identity:*

$$\begin{aligned}(\mathrm{var} * \pi_2) * (\gamma \circ \pi_1, \pi_2) &= \mathrm{var} * (\pi_2 \circ (\gamma \circ \pi_1, \pi_2)) && \text{(pred. associativity)}\\ &= \mathrm{var} * \pi_2 && \text{(property of products)}\end{aligned}$$

- *Preserves composition:*

$$\begin{aligned}&(M * (\pi_1, N/\mathrm{var})) * (\gamma \circ \pi_1, \pi_2)\\ &= M * ((\pi_1, N/\mathrm{var}) \circ (\gamma \circ \pi_1, \pi_2)) && \text{(pred. assoc.)}\\ &= M * ((\pi_1 \circ (\gamma \circ \pi_1, \pi_2)), N/\mathrm{var} \circ (\gamma \circ \pi_1, \pi_2)) && \text{(naturality of } (-,=))\\ &= M * (\gamma \circ \pi_1, N/\mathrm{var} \circ (\gamma \circ \pi_1, \pi_2)) && \text{(property of products)}\\ &= M * (\gamma \circ \pi_1, (N * (\gamma \circ \pi_1, \pi_2))/\mathrm{var}) && \text{(naturality of } -/\mathrm{var})\\ &= M * ((\gamma \circ \pi_1 \circ (\pi_1, (N * (\gamma \circ \pi_1, \pi_2))/\mathrm{var}), \pi_2 \circ (\pi_1, (N * (\gamma \circ \pi_1, \pi_2))/\mathrm{var}))\\ & && \text{(property of products)}\\ &= M * ((\gamma \circ \pi_1, \pi_2) \circ (\pi_1, (N * (\gamma \circ \pi_1, \pi_2))/\mathrm{var})) && \text{(naturality of } (-,=))\\ &= (M * (\gamma \circ \pi_1, \pi_2)) * (\pi_1, (N * (\gamma \circ \pi_1, \pi_2))/\mathrm{var})\end{aligned}$$

Syntactically, all these operations are just substitution, but when we work explicitly categorically with product structure, we're always explicit about when we perform these weakening things.

2 Empty Type

The reason why we introduce these auxiliary unary categories is to give a concise definition of when C-T Structure has an empty type and sum types. To say it in a concise way, let's start with the empty type:

Definition 3. *An* empty type *in a C-T structure S is*

1. *A type 0* $\in S_T$

2. *such that* $\forall \Gamma \in S_C$*, 0 is initial in* Un_Γ

An initial object means that for every other object in the category, there's an unique morphism from zero to it. Thus:

$$\forall A. \exists!\, \mathrm{case}_0 \in Un_\Gamma(0, A) = \mathrm{Tm}A(\Gamma \times \mathrm{sole}0)$$

What we should show is that if we include the empty type in simple type theory, then the syntactic C-T structure $\mathscr{L}$ has an empty type in this sense. That means we need to pick a single type 0 such that for each Γ and for each A, we can construct a term $\Gamma, x : 0 \vdash M : A$ The existence part is exactly as below:

$$\frac{\Gamma, x : 0 \vdash x : 0}{\Gamma, x : 0 \vdash \mathrm{case}_0\, x : A}$$

I.e., we use the elimination form for 0.

Then we need to show that this is *unique* term with one version of the η rule:

$$\frac{\Gamma, x : 0 \vdash M : A}{\Gamma, x : 0 \vdash M = \mathrm{case}_0\, x : A}$$

And this shows the syntactic CT structure $\mathscr{L}$ has an empty type given by the empty type itself.

3 Sum Type

Next, we want to talk about the sum type.

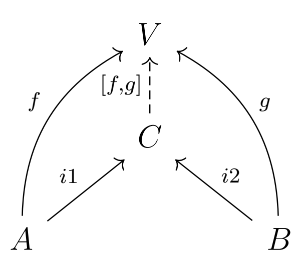
Definition 4. *Let S be C-T structure and Let A,B* $\in S_T$ *An sum type for A, B consists of:*

1. *A type C* $\in S_T$

2. $\forall \Gamma$*, a coproduct structure* $(C, i_1^\Gamma, i_2^\Gamma)$ *for A, B in* Un_Γ*, that is*

$$i_1^\Gamma \in Un_\Gamma(A, C)$$
$$i_2^\Gamma \in Un_\Gamma(B, C)$$

such that we get a unique existence property that for any $f \in \mathrm{Un}_\Gamma AV$ *and* $g \in \mathrm{Un}_\Gamma BV$ *there exists a unique* $[f, g] \in \mathrm{Un}_\Gamma CV$ *that makes the following diagram commute:*



3. *Such that for any* $\gamma \in S_c(\Delta, \Gamma)$,

$$\mathrm{Un}_\gamma(i_1^\Gamma) = i_1^\Delta$$

and

$$\mathrm{Un}_\gamma(i_2^\Gamma) = i_2^\Delta$$

Then, we want to figure out why this make sense from a syntactic point of view. First of all, this type is going to be actually the type $A + B$ in the syntactic CT structure. And we have to exhibit all of the structure, the two injections and the unique existence for each fixed Γ:

$$\Gamma, x : A \vdash i_1(x) : A + B$$
$$\Gamma, x : B \vdash i_2(x) : A + B$$

$$\frac{\Gamma, x : A \vdash M : D \qquad \Gamma, x : B \vdash N : D}{\Gamma, x : A + B \vdash case_+, \{i_1 x \to M \mid i_2 x \to N\} : D}$$

Then, we need to show this actually has the universal property of the co-product. By composing the diagram with i_1:

$$case_+ x \left\{ \begin{array}{l} i_1(x) \to M \\ i_2(x) \to N \end{array} \right\} [i_1(x)/x]$$

which is equal to(by definition):

$$case_+ \, (i_1(x)) \left\{ \begin{array}{l} i_1(x) \to M \\ i_2(x) \to N \end{array} \right.$$

And then we can apply the beta rule for plus to show that is equal to M

$$M[x/x] = M$$

Finally, for the last condition that transport these injections using these functors Un_γ, syntactically, i_1 and i_2 don't depend on the Γ really. So what we need is:

$$\frac{\Gamma, x : A \vdash i_1(x) : A + B \quad \gamma : \Delta \to \Gamma}{\Delta, x : A \vdash i_1^\Gamma(x)[\gamma] = i_1^{(\Delta)xA}(x)}$$

which follows by the definition of substitution.

4 Category with products to C-T Structure

Then let's look at what happens if we have a category with products C, and then we turn it into a C-T structure.

Let $\mathcal{C}$ be a category with finite products. In other word, it has a terminal object and product objects for every A and B. Then we define:

$$(\mathrm{self}\mathcal{C})_C := \mathcal{C}$$

$$(\text{self}\mathcal{C})_T := \mathcal{C}_C$$
$$\text{Tm}\, a\, b := \mathcal{C}(b, a)$$

In this setting, we should also look at what sole is:

$$\mathcal{C}(b,\ \text{sole}\ a) \cong \text{Tm}_{\text{sole}}\, a\, b = \mathcal{C}_1(b, a)$$

$$\text{sole}\ a = a$$

$$\text{var} = \text{id}_a$$

$$f/\text{var} = f$$

And we do get an interesting construction with fixing an object b and inspecting Un_b:

$$\begin{aligned}\text{Un}_b^{self\mathcal{C}}(a, a') &:= \text{Tm}\, a'(b \times \text{sole}\, a)\\ &= \mathcal{C}\,(b \times a, a')\end{aligned}$$

If we pick $b = 1$, the terminal object, we get something equivalent to $\mathcal{C}$, but we see that requiring self$\mathcal{C}$ to have sum types is *stronger* in general than just self$\mathcal{C}$ having coproducts. It corresponds to $\mathcal{C}$ having *distributive* coproducts (see problem set 5). Similarly for the initial object/empty type.

5 Soundness and Completeness Theorems for STT Semantics

The last thing that we want to cover is just to summarize what the soundness and completeness theorem is. And as we've already talked a bit about the soundness theorem here and there throughout. We want to give at least the formulation of the Completeness Theorem.

For simplicity, we fix Σ_0 a set of base types, and work with simple type theory generated from Σ_0 with all connectives $1, \times, 0, +, \Rightarrow$.

Lemma 1. *The syntactic CT structure $\mathscr{L}$ has unit, product, empty, sum and function types.*

Definition 5. *Define the* universal interpretation $\eta : \Sigma_0 \to \mathscr{L}_T$ *to be the inclusion of base types into all types.*

Theorem 1 (Weak Initiality/Soundness and completeness)**.** *For any CT structure S that has unit, product, empty, sum and function types, and any function $i : \Sigma_0 \to S_T$,*

- *(Soundness): We can construct a CT structure homomorphisms*

$$[\![\cdot]\!]^i : \mathscr{L} \to S$$

that

  - *preserves unit, product, empty, sum and function types*

- *and preserves the interpretation in that for every base type* $X \in \Sigma_0$

$$[\![X]\!]^i_T = i(X)$$

- *(Completeness): Furthermore, this homomorphism* $[\![\cdot]\!]^i$ *is* essentially unique*, i.e., if* $F : \mathscr{L} \to S$ *is a CT structure homomorphism that preserves unit, product, empty, sum and function types as well as preserves base types, then there exists a unique natural isomorphism*

$$\alpha : S^{\mathscr{L}_c}_c([\![\cdot]\!]^i_c, F_c)$$

This is weaker than the corresponding theorem we had for Heyting algebras, where the Heyting algebra homomorphism out of the Lindenbaum algebra was unique up to equality. The reason is inherent in the generaliztion from posets to categories: while meets and joins are unique up to equality, universal properties (products, exponentials, etc) are only unique up to unique isomorphism.