Lecture 10: Universal Properties II, Exponentials and Predicators

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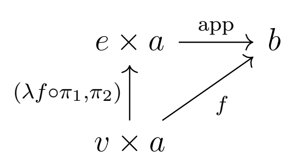
1 Exponential

1.1 Definition

Let $\mathscr{C}$ has binary products. An exponential of $a, b \in \mathscr{C}_0$ is

1. An object $e \in \mathscr{C}_0$ and
2. A morphism $\text{app} : e \times a \to b$ such that for any $f : v \times a \to b$, $\exists! \lambda f : v \to e$ so that the following diagram commutes.

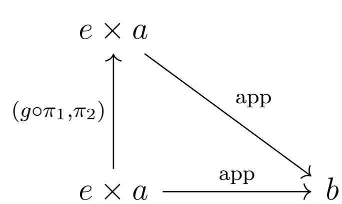


One may make an analogy to the Heyting implication e between a and b where $e \wedge a \leq b$ and for any $x \wedge a \leq b$, $x \leq e$.

1.2 Uniqueness of Exponential

Consider two exponential objects (e, app, λ) and $(e', \text{app}', \lambda')$ of a and b. Then e and e' are isomorphic with the morphisms being $\lambda\text{app}' : e' \to e$ and $\lambda'\text{app} : e \to e'$. By symmetry it suffices to show that $\lambda\text{app}' \circ \lambda'\text{app} = \text{id}_e : e \to e$.

To prove this it is sufficient to show that picking either of $g = \lambda\text{app}' \circ \lambda'\text{app}$ or $g = \text{id}_e$ makes the following diagram commute:



First, to show that id_e makes the diagram commute, it is sufficient to show that $(\mathrm{id}_e \circ \pi_1, \pi_2) = \mathrm{id}_{e\times a}$. To prove this, it is sufficient by the universal property of the *product* to show that

$$\pi_1 \circ (\mathrm{id}_e \circ \pi_1, \pi_2) = \pi_1 \circ \mathrm{id}_e$$

and

$$\pi_2 \circ (\mathrm{id}_e \circ \pi_1, \pi_2) = \pi_2 \circ \mathrm{id}_e$$

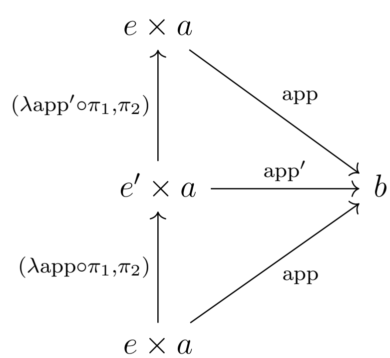
Which follows by definition of the pairing $(-, =)$ operation.

Next, we need to show that

$$\mathrm{app} \circ (\lambda\mathrm{app}' \circ \lambda'\mathrm{app} \circ \pi_1, \pi_2) = \mathrm{app}$$

$$\begin{aligned}
\mathrm{app} \circ (\lambda\mathrm{app}' \circ \lambda'\mathrm{app} \circ \pi_1, \pi_2) &= \mathrm{app} \circ (\lambda\mathrm{app}' \circ \pi_1, \pi_2) \circ (\lambda'\mathrm{app} \circ \pi_1, \pi_2) && \text{(see below)}\\
&= \mathrm{app}' \circ (\lambda'\mathrm{app} \circ \pi_1, \pi_2) && \text{(property of } \lambda\mathrm{app}')\\
&= \mathrm{app} && \text{(property of } \lambda'\mathrm{app})\\
&= \mathrm{app} \circ \mathrm{id}_e
\end{aligned}$$

Besides the unjustified first step, this argument is neatly described by the following diagram:



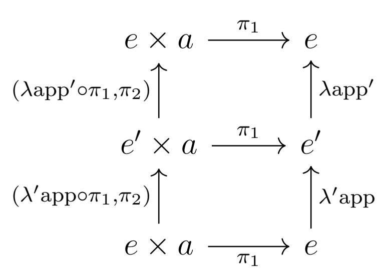
It remains to to show that

$$(\lambda\mathrm{app}' \circ \lambda'\mathrm{app} \circ \pi_1, \pi_2) = (\lambda\mathrm{app}' \circ \pi_1, \pi_2) \circ (\lambda'\mathrm{app} \circ \pi_1, \pi_2) : e \times a \to e \times a$$

By the universal property of a *product* it suffices to show they are equal when applying π_1 and π_2. First,

$$\begin{aligned}
\pi_1 \circ (\lambda\mathrm{app}' \circ \lambda'\mathrm{app} \circ \pi_1, \pi_2) &= \lambda\mathrm{app}' \circ \lambda'\mathrm{app} \circ \pi_1\\
&= \lambda\mathrm{app}' \circ \pi_1 \circ (\lambda'\mathrm{app} \circ \pi_1, \pi_2)\\
&= \pi_1 \circ (\lambda\mathrm{app}' \circ \pi_1, \pi_2) \circ (\lambda'\mathrm{app} \circ \pi_1, \pi_2)
\end{aligned}$$

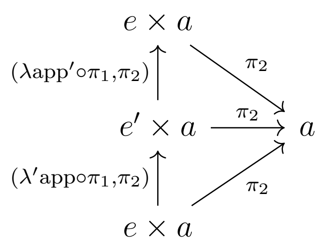
the last two steps are described in the diagram:



Next,

$$\begin{aligned}\pi_2 \circ (\lambda\text{app}' \circ \lambda'\text{app} \circ \pi_1, \pi_2) &= \pi_2 \\ &= \pi_2 \circ (\lambda'\text{app} \circ \pi_1, \pi_2) \\ &= \pi_2 \circ (\lambda\text{app}' \circ \pi_1, \pi_2) \circ (\lambda'\text{app} \circ \pi_1, \pi_2)\end{aligned}$$

where the last two steps are described in the diagram:



Therefore, the exponential object is unique up to isomorphism. Additionally it is unique up to *unique* isomorphism $i : e \to e'$ satisfying $\text{app} \circ (i \circ \pi_1, \pi_2) = \text{app}'$, since this is the unique morphism satisfying the property at all.

1.3 Examples

- In Set, the exponential of set A and B is the set of functions $B^A = \{f : A \to b\}$.
- In Gph, the exponential H^G of graphs G and H can be constructed as

$$\begin{aligned}\left(H^G\right)_v &= H_v^{G_v} \\ \left(H^G\right)_e (f, g) &= \prod_{v \in G_v} H_e(f(v), g(v))\end{aligned}$$

- In Mon, there is no general exponential.

1.4 Free Monoid

For any $A \in \mathsf{Set}$, we have $\texttt{List}A \in \mathsf{Mon}$ defined as the lists of elements in A with concatenation operation. This monoid is called the *free* monoid over A because it satisfies the following property:

1. A morphism $\texttt{single} : A \to |\texttt{List}A|$ that maps $a \in A$ to a singleton list (a) and
2. For any $f : A \to |M|$, $\exists! \bar{f} : \mathsf{Mon}(\texttt{List}A, M)$,such that $f = \texttt{single} \circ |\bar{f}|$.

By a similar argument if we had a different monoid L' with function $s : A \to L'$ such that for any $f : A \to |M|.\exists! \bar{f}' : \mathsf{Mon}(L', M)$ satisfying $f = s \circ |\bar{f}'|$, then we would be able to show that $\texttt{List}A$ is unique up to unique $s/\texttt{single}$-preserving isomorphism $\overline{\texttt{single}}'$

2 Predicators

2.1 Meet and Down Set

For $S \subseteq |P|$, the meet m of S is the greatest lower bound of S, that is,

1. m is a lower bound for S in that $\forall x \in S, m \leq x$
2. m is greater than any other lower bound: $\forall y, (\forall x \in S, y \leq x) \Rightarrow y \leq m$

Define the down set of S as

$$\downarrow S := \{p \in |P| : \forall x \in S, p \leq x\}$$

Then we can equivalently define that m is the meet of S when it is the *greatest* element of $\downarrow S$:

1. First, m is an element of $\downarrow S$: $m \in \downarrow S$
2. Next, it is the greatest element: $x \in \downarrow S \Rightarrow x \leq m$

$\downarrow S$ has the property of being *downward-closed*: $\forall x \in \downarrow S, y \leq x \Rightarrow y \in \downarrow S$.

Then we are able to describe all of our connectives in IPL by saying that they are greatest elements of some downward closed set:

- A top element $\top$ is the greatest element of the entire set $|P|$ (trivially downward closed).
- A binary meet $x \wedge y$ is the greatest element of the downward closed set of lower bounds of x and y: $\{z | z \leq x \wedge z \leq y\}$
- A Heyting implication $x \Rightarrow y$ is the greatest element of the downward closed set $\{z | z \wedge x \leq y\}$

Or, dually, that they are *least* elements of an *upward* closed set:

- A bottom element $\bot$ is the least element of all of $|P|$.
- A join $x \vee y$ is the least element of $\{z | x \leq z \wedge y \leq z\}$

2.2 Predicator

Now we will develop a generalization of downward-closed sets that will allow us to unify all of the different universal properties we've seen so far in the same way that downward-closed sets generalized all connectives in IPL.

We call this notion a *predicator*[1] on the category. We call them predicators as they generalize predicates in a similar way that functors generalize functions.

A predicator P on a category $\mathscr{C}$ consists of

[1] these are more commonly called *presheaves* but then you'd ask what a sheaf is, which won't be relevant until maybe the last week of the course.

1. $\forall a \in \mathscr{C}$, a set $P(a)$
2. An operation $*^{ab} : P(b) \times \mathscr{C}(a, b) \to P(a)$ which satisfies

$$\varphi * \mathrm{id}_b = \varphi$$
$$\varphi * (f \circ g) = (\varphi * f) * g$$

We think of $*$ here as a kind of "composition" operation between elements of the sets $P(b)$ and real morphisms $f \in \mathscr{C}(a, b)$. Then the algebraic identities that we ask to be satisfied are the two of the three category axioms that make sense for the $*$ operation.

To get a feel for predicators, we consider our two extreme special cases: one-object categories, i.e., monoids, and thin categories, i.e., preorders.

If $\mathscr{C}$ has one object $\cdot$, or equivalently, $\mathscr{C}(\cdot, \cdot)$ is a monoid with neutral element e and multiplication $\otimes$, a predicator would be just a single operation $* : P(\cdot) \times \mathscr{C}(\cdot, \cdot) \to \mathscr{C}(\cdot, \cdot)$ satisfying

$$\varphi * e = \varphi$$
$$\varphi * (f \otimes g) = (\varphi * f) * g$$

In this case, the predicator P is precisely an *action* of monoid $\mathscr{C}(\cdot, \cdot)$ on the set $P(\cdot)$. If the monoid is a group, this is called a group action. Analogously, a predicator could be called a category action.

Next consider if $\mathscr{C}$ is a thin category, i.e., a preorder *and* we have a presheaf P where each set $P(a)$ has at most one element ($\forall a \in \mathscr{C}, |P(a)| \leq 1$). Then P is really a kind of predicate on objects of the set, we can think of the predicate as true if $P(a)$ is inhabited and false if it is not. Then the $*$ operation means that if $P(a)$ is inhabited and $b \leq a$ then $P(b)$ is inhabited. Then we see that such a predicator P determines to a downward-closed subset of the objects of $\mathscr{C}$.

A predicator P on $\mathscr{C}$ is **just** the same data as $\mathcal{P} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$. The action on objects gives our $P(a)$ and the functorial action is equivalent to the $*$ operation but in a different order:

$$\mathcal{P}_0(a) = P(a)$$
$$\mathcal{P}_1(f : a \to b)(\varphi \in P(b)) = \varphi * f \in P(a)$$

Then the functoriality laws correspond precisely to our rules for predicators:

$$\mathcal{P}_1(\mathrm{id}_a)(\phi) = \mathrm{id}_{\mathcal{P}_0(a)}(\phi) = \phi = \phi * \mathrm{id}_a$$
$$\mathcal{P}_1(f \circ g)(\phi) = (\mathcal{P}_1(g) \circ \mathcal{P}_1(f))(\phi) = \mathcal{P}_1(g)(\mathcal{P}_1(f)(\phi)) = \mathcal{P}_1(g)(\phi * f) = \phi * f * g$$

2.3 STT Terms as a Predicator

In PS2, we have described the category Ctx of context in STT where the objects are contexts and the morphisms are general substitution. Notice that for any fixed type A:

1. For any context Γ, the terms on it $\mathrm{Term}_A(\Gamma) = \{M | \Gamma \vdash M : A\}$ form a set,

2. We have an operation of substitution into a term that takes a term $M \in \text{Term}_A(\Gamma)$ and a substitution $\gamma : \Delta \to \Gamma$ and gives us

$$M[\gamma] \in \text{Term}_A(\Delta)$$

3. Furthermore, we shows that this satisfies two equations:

$$M[\text{id}_\Gamma] = M$$
$$M[\gamma \circ \delta] = M[\gamma][\delta]$$

Therefore, the terms of any fixed type along with action of substitutions form a predicator on the category of contexts and substitutions.