Problem Set 5

Mar 9, 2022

Homework is due the midnight before class on March 17.

For these two problems, you are asked to construct morphisms in a cartesian closed category and prove they satisfy certain properties. To construct a morphism $A \to B$, you can either

1. Use the abstract category operations in Awodey such as $\epsilon, \tilde{f}, \pi_i, (f, g)$ etc.
2. Define a term $x : A \vdash t : B$ in the simple type theory described in Section 2.8 of Shulman. Then you need to also appropriately interpret composition as substitution, etc. In Shulman, the term syntax is on page 135 $\beta\eta$ equations for $0, +, 1, \times$ are on page 114 and $\beta\eta$ for $\Rightarrow$ are on page 139.

Problem 1 Algebra of Exponentials

At this point we have developed quite the "algebra" of objects in cartesian closed categories: in addition to addition $A + B, 0$ and multiplication $A \times B, 1$ we have now added exponentiation B^A.

In a cartesian closed category, many of the familiar rules of arithmetic with exponentiation are valid if we represent equality as *isomorphism*:

$$C^{A \times B} \cong (C^A)^B \qquad C^1 \cong C$$

$$C^{A+B} \cong C^A \times C^B \qquad C^0 \cong 1$$

1. For each of the above isomorphisms $A \cong B$, construct morphisms $\mathrm{to} : A \to B$ and $\mathrm{fro} : B \to A$ that are valid in an arbitrary cartesian closed category.
2. Prove that the morphisms you defined for $C^{A+B} \cong C^A \times C^B$ are an isomorphism, that is $\mathrm{to} \circ \mathrm{fro} = \mathrm{id}_B$ and $\mathrm{fro} \circ \mathrm{to} = \mathrm{id}_A$. You do not need to prove the others form an isomorphism[1]

. . . .

[1]but it will be impossible to construct morphisms of the given type that *aren't* isomorphisms!

Problem 2 Lawvere's Fixed Point Theorem and Recursive Functions

There are various "diagonalization arguments" employed in logic, set theory and computer science such as the proofs of Gödel's incompleteness theorem, Cantor's theorem and Turing/Rice's theorem to prove that some construction is impossible.

F. William Lawvere showed that we can abstract over the reasoning in these different proofs by proving a *fixed point* theorem that is valid in an arbitrary cartesian closed category. We will prove the following slightly simplified version of *Lawvere's Fixed Point Theorem*.

Let X, D be objects in a cartesian closed category. If there is a section-retraction pair $(s : D^X \to X, r : X \to D^X)$ then there is a morphism $\mathrm{fix} : D^D \to D$ that is a *fixed point combinator* in that for any $f : \Gamma \to D^D$,

$$\mathrm{fix} \circ f = \epsilon \circ (f, \mathrm{fix} \circ f)$$

In type theoretic notation, this is a term

$$f : (D \Rightarrow D) \vdash \mathrm{fix}(f) : D$$

satisfying

$$\mathrm{fix}(f) = f(\mathrm{fix}(f))$$

1. Instantiate the theorem to the category of sets with $D = \{\mathrm{true}, \mathrm{false}\}$ and X an arbitrary set to prove *Cantor's theorem*: there is no surjective function from X to its power set $\mathcal{P}(X)$.

2. Given the section-retraction pair (s, r), construct fix and prove it is a fixed point combinator.

This theorem is not just for proving contradictions! You may recognize fix as a typed version of the Y-combinator, which can be used to encode recursion in programming languages.

As an example, given a base type N for natural numbers and function symbols[2] $\mathrm{sub1} : N \to 1 + N$ and $\mathrm{times} : N, N \to N$ and $\mathrm{one} : \cdot \to N$, we can use our fixpoint combinator to implement a factorial function:

$$\mathrm{fix}(\lambda \mathrm{fac}.\lambda n.\mathrm{match}_{+}(\mathrm{sub1}(n))\{\sigma_1 x_1.\mathrm{one}\}\{\sigma_2 m.\mathrm{times}(n, \mathrm{fac}\, m)\}$$

....

[2]We interpret sub1(n) as returning $\sigma_1()$ if $n = 0$ and otherwise $\sigma_2(n-1)$