Problem Set 6: Adjunctions and Algebras

Released: November 13, 2025
Due: Novemmber 25, 2025, 11:59pm

Submit your solutions to this homework on Canvas alone or in a group of 2. Your solutions must be submitted in pdf produced using LaTeX.

Definition 1 (Algebraic Theory of Immutable State). *The algebraic theory of an immutable boolean state T_{imm} consists of one binary operation*

- *"read": $r(x, y)$*

and two laws

- *Constancy law $r(x, x) = x$*
- *Diagonal law $r(r(x_{00}, x_{01}), r(x_{10}, x_{11})) = r(x_{00}, x_{11})$*

Definition 2 (Algebraic Theory of Mutable State). *The algebraic theory of mutable boolean state T_{mut} consists of two operations*

- *binary operation "read": $r(x, y)$*
- *two unary operations "set 0" $s_0(x)$ and "set 1" $s_1(x)$*

Subject to the following laws

- *Diagonal law: $r(r(x_{00}, x_{01}), r(x_{10}, x_{11})) = r(x_{00}, x_{11})$*
- *Read-set: $r(s_0(x), s_1(x)) = x$*
- *Set-read: $s_i(r(x_0, x_1)) = s_i(x_i)$*
- *Set-set: $s_i(s_j(x)) = s_j(x)$*

Definition 3 (Free Algebra). *A free T-algebra on a set A for an algebraic theory T consists of the following data:*

- *An algebra FA*
- *A function $\eta : A \to UFA$*
- *Such that for every algebra Y pre-composition with η is a bijection T-Alg$(FA, Y) \to$ Set(A, UB) from homomorphisms out of the free algebra on A to functions into the underlying set of the algebra B*

Problem 1 Relating properties of adjoint functors and their (co)-units

Prove Lemma 4.5.13 from Riehl's *Category Theory in Context*, page 140. Explain precisely how the case for unit follows from the case for co-unit (or vice-versa) by duality.

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Problem 2 Normal forms for computational algebras

1. Let A be a set. Define a T_{imm}-algebra structure whose underlying set is A^2 and show that it is the free T_{imm}-algebra on A.
2. Show that for any T_{mut}-algebra, the read operation satisfies the axioms of a T_{imm}-algebra.
3. Let A be a set. Define a T_{mut}-algebra structure whose underlying set is $(2 \times A)^2$ and show that it is the free T_{mut}-algebra on A.

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