Problem Set 3: Functoriality and Naturality

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Submit your solutions to this homework on Canvas alone or in a group of 2. Your solutions must be submitted in pdf produced using LaTeX.

Problem 1 Bifunctors

Functors provide a notion of single-argument morphism between categories. A *bifunctor* from $\mathcal{C}$ and $\mathcal{D}$ to $\mathcal{E}$ is a notion of "functor of two arguments". In this problem we will show that various ways of defining a bifunctor are equivalent.

- A *jointly functorial* bifunctor $F : \mathcal{C}, \mathcal{D} \to \mathcal{E}$ consists of
  1. An action on objects $F_0 : \mathcal{C}_0 \times \mathcal{D}_0 \to \mathcal{E}_0$
  2. A joint action on morphisms, which for every $A, A' \in \mathcal{C}_0$ and $B, B' \in \mathcal{D}_0$ gives a function $F_1 : \mathcal{C}_1(A, A') \times \mathcal{D}_1(B, B') \to \mathcal{E}_1(F_0(A, B), F_0(A', B'))$
  3. satisfying joint functoriality laws that $F_1(\mathrm{id}_A, \mathrm{id}_B) = \mathrm{id}_{F_0(A,B)}$ and $F_1(f \circ f', g \circ g') = F_1(f, g) \circ F_1(f', g')$
- A *separately functorial* bifunctor $F : \mathcal{C}, \mathcal{D} \to \mathcal{E}$ consists of
  1. An action on objects $F_0 : \mathcal{C}_0 \times \mathcal{D}_0 \to \mathcal{E}_0$
  2. A *left* action on morphisms, which for every $A, A' \in \mathcal{C}_0$ and $B \in \mathcal{D}_0$ gives a function $F_l : \mathcal{C}_1(A, A') \to \mathcal{E}_1(F_0(A, B), F_0(A', B))$
  3. A *right* action on morhpisms, which for every $A \in \mathcal{C}_0$ and $B, B' \in \mathcal{D}_0$ gives a function $F_r : \mathcal{D}_1(B, B') \to \mathcal{E}_1(F_0(A, B), F_0(A, B'))$
  4. Satisfying left functoriality laws $F_l(\mathrm{id}_A) = \mathrm{id}_{F_0(A,B)}$ and $F_l(f \circ f') = F_l(f) \circ F_l(f')$
  5. Satisfying right functoriality laws $F_r(\mathrm{id}_B) = \mathrm{id}_{F_0(A,B)}$ and $F_r(g \circ g') = F_r(g) \circ F_r(g')$
  6. Satisfying a commutative law $F_l(f) \circ F_r(g) = F_r(g) \circ F_l(f)$.

1. Construct a bijection between separately functorial and jointly functorial bifunctors $\mathcal{C}, \mathcal{D} \to \mathcal{E}$.

2. Construct a bijection between jointly functorial bifunctors $\mathcal{C}, \mathcal{D} \to \mathcal{E}$ and functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$.

3. Construct a bijection between separately functorial bifunctors $\mathcal{C}, \mathcal{D} \to \mathcal{E}$ and functors $\mathcal{C} \to \mathcal{E}^{\mathcal{D}}$, where $\mathcal{E}^{\mathcal{D}}$ is the category of functors from $\mathcal{D}$ to $\mathcal{E}$ with natural transformations as morphisms.

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Problem 2 Product Functor

Let $\mathcal{C}$ be a category such that for every pair of objects $A, B \in \mathcal{C}$, we have a specified product $(A \times B, \pi_1 : \mathcal{C}(A \times B, A), \pi_2 : \mathcal{C}(A \times B, B))$.

1. Show that taking binary products defines a functor $\times : (\mathcal{C} \times \mathcal{C}) \to \mathcal{C}$. That is, show that if we define $\times$ on objects such that $a \times b$ is a product of a and b (with projections $\pi_1 : a \times b \to a$ and $\pi_2 : a \times b \to b$), then you can extend the definition to a functorial action on morphisms.

2. Let $\Pi_1 : \mathcal{C}^2 \to \mathcal{C}$ be the functor that projects out the first component of $\mathcal{C}^2$. Prove that π_1 defines a natural transformation from $\times$ to Π_1. Symmetrically, π_2 is also natural.

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Problem 3 Theorems for Free, Naturally

The naturality property of a natural transformation is such a strong condition that sometimes we can characterize all natural transformations between two fixed functors, and in many examples there are only finitely many.

This has direct applications to programming. The reason is that in a pure polymorphic functional language, given type constructors F and G that are functorial, all functions $F(X) \to G(X)$ that are polymorphic in X denote natural transformations! Phil Wadler, building on John Reynold's theory of parametricity called these "theorems for free": just from the type of a polymorphic function, the naturality property gives you properties that hold for every function of that type (Reynolds [1983], Wadler [1989]).

1. Define a natural transformation from $\mathrm{id}_{\mathrm{Set}}$ to $\mathrm{id}_{\mathrm{Set}}$ and prove that it is the only such natural transformation.

2. Let $\times : \mathrm{Set} \times \mathrm{Set} \to \mathrm{Set}$ be the functor you deifned in the previous problem and let $\times' : \mathrm{Set} \times \mathrm{Set} \to \mathrm{Set}$ be the functor with the arguments swapped $A \times' B = B \times A$.

   Define a natural transformation from $\times$ to $\times'$ and show that it is the only such natural transformation.

3. Recall the category of pointed sets Set_* is defined as follows:

   - Objects are pairs of a set X and a "basepoint" $x_0 \in X$.
   - A morphism from (X, x_0) to (Y, y_0) is a *base-point-preserving* function, i.e., a function $f : X \to Y$ such that $f(x_0) = y_0$. Identity and composition are simply identity and composition of functions.

   Define two different natural transformations from $\mathrm{id}_{\mathrm{Set}_*}$ to $\mathrm{id}_{\mathrm{Set}_*}$ and prove that these are the only two such natural transformations.

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References

John C. Reynolds. Types, abstraction and parametric polymorphism. In R. E. A. Mason, editor, *Information Processing 83, Proceedings of the IFIP 9th World Computer Congress, Paris, France, September 19-23, 1983*, pages 513–523. North-Holland/IFIP, 1983.

Philip Wadler. Theorems for free! In Joseph E. Stoy, editor, *Proceedings of the fourth international conference on Functional programming languages and computer architecture, FPCA 1989, London, UK, September 11-13, 1989*, pages 347–359. ACM, 1989. doi: 10.1145/99370.99404. URL `https://doi.org/10.1145/99370.99404`.