# Lecture 11: Universal Properties III

Lecturer: Max S. New
Scribe: Yuchen Jiang

October 1, 2025

## 1 Presheaves and Yoneda's Lemma

Recap: definition of a presheaf.

**Definition 1.** A presheaf $\mathcal{P}$ on a category $\mathcal{C}$ defines

- $\forall X \in C_0$, a set $\mathcal{P}(X)$
- $\forall X, Y \in C_0$ and $f : Y \to X$, an action $\circ_{\mathcal{P}}$

  $$\frac{p : \mathcal{P}(X) \qquad f : Y \to X}{p \circ_{\mathcal{P}} f : \mathcal{P}(Y)}$$

  and the action should look like composition, i.e. the identity and composition of morphisms in $\mathcal{C}$ should be preserved.

  $$p \circ_{\mathcal{P}} \mathrm{id}_X = p$$
  $$p \circ_{\mathcal{P}} (g \circ_{\mathcal{C}} f) = (p \circ_{\mathcal{P}} g) \circ_{\mathcal{P}} f$$

*Remark.* The definition above can be interpreted as a functor: a presheaf $\mathcal{P}$ on a category $\mathcal{C}$ is a functor $P : \mathcal{C}^{op} \to \mathrm{Set}$.

As a result, the right notion of morphism between presheaves is a natural transformation. Let $\mathcal{P}$ and $\mathcal{Q}$ be presheaves on $\mathcal{C}$, then a natural transformation $\alpha : \mathcal{P} \Rightarrow \mathcal{Q}$ is a family of morphisms $\alpha_X : \mathcal{P}(X) \to \mathcal{Q}(X)$ for all $X \in \mathcal{C}_0$ s.t.

$$\forall f : Y \to X \text{ and } p : \mathcal{P}(X), \alpha_Y(p \circ_{\mathcal{P}} f) = \alpha_X(p) \circ_{\mathcal{Q}} f$$

We also talked about the analogy between sets, preorders, and categories.

| Set | Preorder | Category |
|---|---|---|
| $=$ | $\leq$ | $\to$ |
| Functions | Monotone Functions | Functors |
| Subsets / Predicates | Downwards-Closed Subsets | Presheaves |

Now let's talk about how the Yoneda's Lemma looks like in these different contexts.

## Yoneda's Lemma for Sets

For any set $X$, there exists its power set $\mathscr{P}X$ and a function

$$X \xrightarrow{\{-\}} \mathscr{P}X$$

which is the singleton set. Yoneda's Lemma for Sets simply states that $\forall X, \forall P \subseteq X$,

$$\{x\} = P \text{ iff } x \in P \text{ and } \forall y \in P,\ y = x$$

We can instead think of $\mathscr{P}$ as a function $X \to 2$, i.e. a predicate, then the following holds:

$$\exists! x.P(x) := \{x\} = \{y \mid P(y)\}$$

## Yoneda's Lemma for Preorders

We've already seen the baby Yoneda's Lemma for preorders. $\forall x, S \subseteq_{\text{down}} P$, the principle downset $\downarrow x \subseteq S$ iff $x \in S$.

## Yoneda's Lemma for Categories

Given a category $\mathcal{C}$, we can define a presheaf よ for each object $X \in \mathcal{C}_0$ as follows:

- $よX : \mathcal{C}^{op} \to \text{Set}$
- At some object $Y$, $(よX)(Y) := \mathcal{C}(Y, X)$
- Since $(よX)(Y)$ is a set of morphisms, we can specify the action of $よX$ as

$$\frac{f \in (よX)(Y) \qquad g \in \mathcal{C}(Z, Y)}{f \circ_{よ} g := f \circ g}$$

Then the Yoneda's Lemma is a characterization of the representable presheaves, which is called the Yoneda embedding. It's a universal property of the presheaf $よX$ as an object of the category of presheaves.

Stepping back for a second, we can ask a question: what's an obvious element of the presheaf $よX$? Well we can certainly think of the identity morphism:

$$\text{id}_X : (よX)(X)$$

In fact, that's the only element that is guaranteed to be in the presheaf. What the Yoneda lemma tells us is that the presheaf $よX$ is freely generated from this one element $\text{id}_X$.

**Lemma 1.** *$\forall p : \mathcal{P}(X)$, there exists a unique natural transformation $[p] : よX \Rightarrow \mathcal{P}$ s.t. $[p]_X(\text{id}_X) = p$*

*Proof.* Given $p : \mathcal{P}(X)$, we can define $[\,p\,] : よX \Rightarrow \mathcal{P}$ as follows: $\forall Y \in \mathcal{C}_0$,

$$[\,p\,]_Y(f) := p \circ_{\mathcal{P}} f$$

Then we need to show that it's natural and unique.

By natural we mean $[\,p\,]_X(\mathrm{id}_X) = p$ and $[\,p\,](g \circ f) = [\,p\,](g) \circ_{\mathcal{P}} f$. They both follow from the definition of $[\,p\,]$.

By unique we mean for any two natural transformations $\alpha, \beta : よX \Rightarrow \mathcal{P}$ s.t. $\alpha_X(\mathrm{id}_X) = p = \beta_X(\mathrm{id}_X)$, we have $\alpha = \beta$. We want to show that $\alpha_Y(f) = \beta_Y(f)$ for all $Y \in \mathcal{C}_0$ and $f : Y \to X$. Since

$$\alpha_Y(f) = \alpha_Y(\mathrm{id}_X \circ f) = \alpha_X(\mathrm{id}_X) \circ_{\mathcal{P}} f = p \circ_{\mathcal{P}} f$$

and similarly for $\beta$, we get $\alpha_Y(f) = \beta_Y(f)$. Therefore $\alpha = \beta$. □

# 2 Universal Elements of Presheaves

We can further generalize $よX$ in the Yoneda Embedding to be an arbitrary presheaf $\mathcal{P}$ with the concept of universal elements of presheaves. A universal $X$-element of presheaf $\mathcal{P}$ is an element $\eta : \mathcal{P}(X)$ s.t. $\forall q : \mathcal{Q}(X)$, there exists a unique $[\,q\,] : \mathcal{P} \Rightarrow \mathcal{Q}$ satisfying

$$[\,q\,]_X(\eta) = q$$

Universal elements are unique up to unique isomorphism.

**Theorem 1.** *Given a presheaf $\mathcal{P}$ and an universal $X$-element $\eta : \mathcal{P}(X)$ and another presheaf $\mathcal{Q}$ and an universal $X$-element $\epsilon : \mathcal{Q}(X)$, there exists a unique natural isomorphism $i : \mathcal{P} \Rightarrow \mathcal{Q}$ s.t. $i_X(\eta) = \epsilon$.*

*Proof.* We simply define $i := [\,\epsilon\,] : \mathcal{P} \Rightarrow \mathcal{Q}$ and its inverse $i^{-1} := [\,\eta\,] : \mathcal{Q} \Rightarrow \mathcal{P}$. Then we verify that $i$ is indeed a natural isomorphism. We want to show (and similarly for $i \circ i^{-1}$):

$$i^{-1} \circ i = \mathrm{id} : \mathcal{P} \Rightarrow \mathcal{P}$$

which means $i^{-1}(i(\eta)) = \eta$, namely $[\,\eta\,]([\,\epsilon\,](\eta)) = \eta$. By definition, $[\,\epsilon\,](\eta) = \epsilon$, and $[\,\eta\,](\epsilon) = \eta$. Therefore the above equation holds, and similarly for the other direction. □

**Corollary 1.** *If $\eta_X : \mathcal{P}(X)$ is a universal element, then*

$$[\,\eta_X\,] : よX \xrightarrow{\sim} \mathcal{P}$$

We denote natural isomorphism by $\xrightarrow{\sim}$.

A second part of the Yoneda's Lemma:

**Lemma 2.** *The universal element of a presheaf $\mathcal{P}$ at object $X$ is isomorphic to the natural isomorphism between the Yoneda embedding $よX$ and $\mathcal{P}$.*

$$\text{UnivElt } \mathcal{P}(X) \;\cong\; \text{NatIso } よX \ \mathcal{P}$$

This part of the lemma means that we can find the universal element if we know the natural isomorphism $i : よX \xrightarrow{\sim} \mathcal{P}$, namely $i_X(\mathrm{id}_X) : \mathcal{P}(X)$ is universal.

*Proof.* Let $q : \mathcal{Q}(X)$ be an arbitrary element of $\mathcal{Q}$. We want to show that there exists a unique natural transformation $[\,q\,] : よX \Rightarrow \mathcal{Q}$ s.t. $[\,q\,]_X(i_X(\mathrm{id})) = q$.

We know that $i^{-1}(p) : (よX)(Y)$, namely $\mathcal{C}(Y, X)$, which can be composed with $q : \mathcal{Q}(X)$ so that we can define $[\,q\,]_Y(p) := q \circ i^{-1}(p) : \mathcal{Q}(Y)$. Therefore the following holds:

$$[\,q\,]_Y(i(\mathrm{id})) = q \circ i^{-1}(i(\mathrm{id})) = q$$

which is exactly what we want to show. □

What we've just shown hints to an elegant construction of the universal element from the natural isomorphism $i : よX \xrightarrow{\sim} \mathcal{P}$. For any presheaf $\mathcal{Q}$, we can construct the natural transformation $\alpha : \mathcal{P} \Rightarrow \mathcal{Q}$ by composing $i^{-1} : \mathcal{P} \Rightarrow よX$ and $[\,q\,] : よX \Rightarrow \mathcal{Q}$:

$$\mathcal{P} \xrightarrow{\sim} よX \Rightarrow \mathcal{Q}$$

which concludes that universal elements are isomorphic to natural isomorphisms. From now on, we shall **define all universal properties in terms of natural isomorphisms** NatIso $よX$ $\mathcal{P}$ **with a clever choice of** $\mathcal{P}$.

# 3 Universal Properties, Revisited

All instances of universal properties that we've seen so far can be formulated in terms of the definition above.

## Terminal Object

A terminal object is an object 1 s.t. for any object $X$, there exists a unique morphism $! : X \overset{\exists!}{\to} 1$. This definition can be rephrased as

$$X \overset{\exists!}{\to} 1 \;\cong\; \text{UnivElt } \mathcal{P}(1)$$

where $\mathcal{P} : \mathcal{C}^{op} \to \mathrm{Set}$ is the presheaf that sends every object to the singleton set (the 1-element set). Namely,

$$\mathcal{P}(X) = \{*\}$$

for all $X \in \mathcal{C}_0$. In fact, we should give $\mathcal{P}$ a name: TermPsh.

What does it mean to be a universal element of TermPsh?

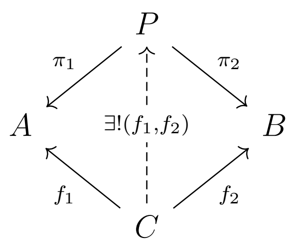
- An element at the terminal object: $* : \mathrm{TermPsh}(1)$
- The action of the element is $* \circ f = *$ for all $f : X \to 1$.
- The element $*$ is universal, namely we can define a natural isomorphism $[\,*\,] : よ1 \Rightarrow \mathrm{TermPsh}$ s.t.

  $$\mathcal{C}(X, 1) \xrightarrow{\sim} \{*\}$$

  where $f \mapsto * \circ f$ for all $f : X \to 1$.

## Product

Given two objects $A, B$ in $\mathcal{C}$, a product is an object $P$ s.t. for any object $C$ that has morphisms to $A$ and $B$, there exists a unique morphism $(f_1, f_2) : C \to P$ s.t. the following diagram commutes:




$\cong$ UnivElt $\text{ProdPsh}(A, B)(P)$

We want to show that all data in the diagram is completely determined by the universal element of $\text{ProdPsh}(A, B)(P)$.

First, we define the presheaf $\text{ProdPsh}(A, B)$ as follows:

- The element at $C \in \mathcal{C}_0$ is defined as $\text{ProdPsh}(A, B)(C) := \mathcal{C}(C, A) \times \mathcal{C}(C, B)$.
- The action $\circ_{\text{ProdPsh}(A,B)}$ is defined as

$$\frac{(f_1, f_2) : \text{ProdPsh}(A, B)(C) \qquad g : \mathcal{C}(D, C)}{(f_1, f_2) \circ_{\text{ProdPsh}(A,B)} g := (f_1 \circ g, f_2 \circ g)}$$

And then we define the universal element $\eta$ of $\text{ProdPsh}(A, B)$ as follows:

- $\eta : \text{ProdPsh}(A, B)(P)$ is defined as $(\pi_1, \pi_2)$ where $\pi_1 : \mathcal{C}(P, A)$ and $\pi_2 : \mathcal{C}(P, B)$ are the projections out of the product $P$.
- We can check that $[\eta] : \text{よ} P \xrightarrow{\sim} \text{ProdPsh}(A, B)$ is a natural isomorphism, which means given $f : \mathcal{C}(C, P)$ for any object $C$, we send it through the natural isomorphism as $f \mapsto (\pi_1, \pi_2) \circ f$, which is exactly $f \mapsto (\pi_1 \circ f, \pi_2 \circ f)$. As a result, we can rewrite $f$ to be a pair $(f_1, f_2) : \text{ProdPsh}(A, B)(C)$.
- Moreover, given $(f_1, f_2) : \text{ProdPsh}(A, B)(C)$, we can take the inverse of the natural isomorphism $[\eta]^{-1} : \text{ProdPsh}(A, B) \xrightarrow{\sim} \text{よ} P$ to get a morphism $(f_1, f_2) : \mathcal{C}(C, P)$. We can then conpress the fact that the diagram commutes into a single equation:

$$(\pi_1 \circ (f_1, f_2), \pi_2 \circ (f_1, f_2)) = (f_1, f_2)$$

  which corresponds awfully well with the $\beta$-laws of the product.
- Similarly, if we start with any object $C$ instead of fixing one, we can get the $\eta$-laws by saying that all morphisms from $C$ to $P$ are the same morphism.

We may also formulate the universal property in terms of the natural isomorphism. Taking product as an example, we have

$$\mathcal{C}(C, P) \cong \mathcal{C}(C, A) \times \mathcal{C}(C, B)$$
$$\text{よ} P \cong \text{ProdPsh}(A, B)$$

## Initial Object

An initial object is an object 0 s.t. for any object $X$, there exists a unique morphism $¡ : 0 \overset{\exists!}{\to} X$. We may attempt to formulate the corresponding presheaf EmpPsh as

$$\text{EmpPsh}(X) := \varnothing$$

However, there are presheaves that are not representable, and the empty presheaf is one of them, meaning that EmpPsh is not representable. Generally speaking, the right-hand universal properties like the terminal object and the products talk about maps into the object, defining morphisms into the object. But the initial object is a left-hand universal property that talks about maps out of the object. As a result, the initial object can only be defined by the terminal object on the opposite category.

It's funny to think about what a presheaf $\mathcal{P}$ looks like when it's on an opposite category $\mathcal{C}^{op}$:

$$\text{Presheaf on } \mathcal{C}^{op} \cong (\mathcal{C}^{op})^{op} \to \text{Set} \cong \mathcal{C} \to \text{Set}$$

It's called a contravariant presheaf, and instead of defining the action $p \circ f$ for a presheaf, we define the action $f \circ p$ for a contravariant presheaf.

As for the initial object in $\mathcal{C}$, we can just define it as the terminal object in $\mathcal{C}^{op}$, defined as $\text{TermPsh}^{\mathcal{C}^{op}}$.

## Coproduct

Similarly, the coproduct can be defined as the product on the opposite category.

$$\text{CoprodPsh}(A, B)^{\mathcal{C}} \cong \text{ProdPsh}^{\mathcal{C}^{op}}(A, B)$$

And if we expand the definition we'll get:

$$\begin{aligned} \text{ProdPsh}^{\mathcal{C}^{op}}(A, B)(C) &:= \mathcal{C}^{op}(C, A) \times \mathcal{C}^{op}(C, B) \\ &= \mathcal{C}(A, C) \times \mathcal{C}(B, C) \end{aligned}$$

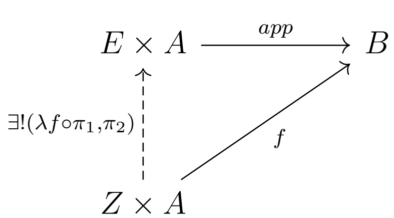
There is also the notion of a sum presheaf:

$$\text{SumPsh}(A, B)(C) := \mathcal{C}(C, A) + \mathcal{C}(C, B)$$

But it's almost never representable.

## Exponential

Given two objects $A, B$, an exponential object $E$ satisfies




$\cong$ UnivElt ExpPsh$(A, B)(E)$

where ExpPsh$(A, B)$ is defined as

- Elements $\text{ExpPsh}(A, B)(C) := \mathcal{C}(C \times A, B)$
- Action

$$\frac{f : \mathcal{C}(C \times A, B) \qquad g : \mathcal{C}(D, C)}{f \circ_{\text{ExpPsh}(A,B)} g : \mathcal{C}(D \times A, B)}$$

  defined as $f \circ_{\text{ExpPsh}(A,B)} g := f \circ (g \circ \pi_1, \pi_2)$.

- ...and the action preserves the identity and composition.

Let's look at two more easier examples.

## Graph Coloring

Given a graph $G$, the $K$-coloring($G$) is a function $\chi : G.v \to [K]$ from vertices to a set of $K$ elements (colors), s.t. if vertices $g \sim h$ are adjacent, then $\chi(g) \neq \chi(h)$. Then an interesting question arises: Is the presheaf $\chi$ representable? (Is there a graph $G$ with a universal $K$-coloring?) That is to say, can we find a graph $\lceil K \rceil$ such that $\chi : K\text{-coloring}(\lceil K \rceil)$ is the universal $K$-coloring?

$$G \xrightarrow{\varphi} \lceil K \rceil \xrightarrow{\chi} [K]$$

In other words, can we find a graph $\lceil K \rceil$ such that for any graph $G$, the following natural isomorphism holds:

$$\text{GraphHom}(G, \lceil K \rceil) \cong K\text{-coloring}(G)$$

The answer is we can define $\lceil K \rceil$ as a complete graph on $K$ vertices. All vertices are connected to each other, except for itself. In this way, each vertex represents a unique color, and all colors can only have neighbors of different colors. Defining the graph homomorphism from $G$ to $\lceil K \rceil$ is then the same as defining a color-assignment function.

## Subobject Classifier

Revisiting the Powerset Functor $\mathscr{P} : \text{Set}^{op} \Rightarrow \text{Set}$. The element $\mathscr{P}(X)$ is the powerset of $X$. Given $f : X \to Y$, we can define $f^{-1} : \mathscr{P}(Y) \to \mathscr{P}(X)$ as

$$f^{-1}(S) := \{x \in X \mid f(x) \in S\}$$

Then what would the universal element of $\mathscr{P}$ be? Suppose we call it $\eta : \mathscr{P}(A)$, and it being universal means functions $X \to A$ should be isomorphic to the subsets of $X$:

$$X \to A \cong \mathscr{P}(X)$$

The only choice of $A$ that satisfies this is $A = 2$, the two-element set $\{0, 1\}$. We can conclude that $X \to 2 \ \cong\ \mathscr{P}(X)$, namely よ$2 \ \cong\ \mathscr{P}$.

If we name the universal element $\eta : \mathscr{P}(2)$, then the natural isomorphism $[\eta] : (X \to 2) \xrightarrow{\sim} \mathscr{P}(X)$ is defined as

$$[\eta]_X(f) := \{x \mid f(x) = 1\}$$

where $f(x) = 1$ means $f(x) \in \{1\}$.

This universal property in topos theory is called "Subobject Classifier", a generalization of the predicate to arbitrary categories.

## 3.1 Universal Properties are Essentially Unique

Finally, we prove one more theorem about universal properties: they are unique up to unique isomorphism. We approach this theorem in two steps.

First, we ask: is $よ : \mathcal{C} \to \text{Psh}(\mathcal{C})$ a functor? We've been defining how $よ$ acts at objects. Now at least we can extend this operation to be a functor by defining how $よ$ acts at morphisms.

Given $f : X \to Y$, we can define $よf : よX \Rightarrow よY$ as follows:

$$(よf)_Z(g : よ(X)(Z)) : よ(Y)(Z)$$
$$(よf)_Z(g) := f \circ g$$

which can be concluded by $よf = f \circ -$.

But to complete our goal of showing that $よ$ is functorial ($よ$ is a functor), we need to show that firstly, $よf$ is natural in the choice of $g$ (since morphisms of a functor $よ$ are natural transformations); secondly, $よf$ preserves the identity and composition.

The naturality of $よf$:

$$f \circ (g \circ h) = (よf)_Z(g \circ h) = ((よf)_Z(g)) \circ h = (f \circ g) \circ h$$

$よf$ preserves the identity and composition

$$(よ\text{id})(g) = \text{id} \circ g = g$$
$$(f \circ g) \circ h = (よ(f \circ g))(h) = (よf)((よg)(h)) = f \circ (g \circ h)$$

Now that we've established that $よ$ is a functor, the second step is to show that:

**Theorem 2.** $よ$ *is fully faithful.*