Lecture 8: Equivalence of Categories

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Topics: equivalence of Categories

1 Motivating Example

Two categories are introduced as motivating examples.

Definition 1. The category of partial functions Par is defined as follows:

- $\mathrm{Par}_0 :=$ Sets, the set of all sets;
- $\mathrm{Par}_1(X, Y) := X \rightharpoonup Y$, all partial functions from X to Y;
- $id_X(x) := x$ for all $x \in X$;
- Composition is defined in the same way as total functions.

Definition 2. The category of pointed sets Set_* is defined as follows:

- $(\mathrm{Set}_*)_0 := (X : \mathrm{Set}) \times X$, where the second element in the pair is a distinguished base point of X. An object in the category Set_* is denoted by $X \ni x_0$ where X is the set and $x_0 \in X$ is the distinguished element.
- $(\mathrm{Set}_*)_1(X \ni x_0, Y \ni y_0) := \{f : X \to Y \mid f(x_0) = y_0\}$, i.e., the set of base-point preserving functions

Then we examine the relationship between the two categories – if they are isomorphic, or how closely are they related. We first define the following function $- \uplus \{\mathrm{err}\} : \mathrm{Par} \to \mathrm{Set}_*$ where:

- For $X \in \mathrm{Par}_0$, $X \uplus \{\mathrm{err}\} := X \uplus \{\mathrm{err}\} \ni \sigma_1\{\mathrm{err}\}$;
- For $f : X \rightharpoonup Y \in \mathrm{Par}_1(X, Y)$, the corresponding morphism in Set_*, $f \uplus \{\mathrm{err}\} : X \uplus \{\mathrm{err}\} \to Y \uplus \{\mathrm{err}\}$ is defined as
  - $(f \uplus \{\mathrm{err}\})(\sigma_0(x)) = \sigma_0(y)$ if $f(x) = y$;
  - $(f \uplus \{\mathrm{err}\})(\sigma_0(x)) = \sigma_1(\mathrm{err})$ if $f(x)$ is undefined;

- $(f \uplus \{\text{err}\})(\sigma_1(\text{err})) = \sigma_1(\text{err})$.

$- \uplus \{\text{err}\}$ can be proved to be a functor.

The function in the other direction can also be defined remove : $\text{Set}_* \to \text{Par}$ where

- For $(X \ni x_0) \in (\text{Set}_*)_0$, $\text{remove}(X \ni x_0) := X - \{x_0\}$;
- For $f : (X \ni x_0) \to (Y \ni y_0)$, the resulting morphism is defined as follows:
  - $\text{remove}(f : (X \ni x_0) \to (Y \ni y_0))(x)$ is undefined if $f(x) = y_0$;
  - $\text{remove}(f : (X \ni x_0) \to (Y \ni y_0))(x) = y$ if $f(x) = y \neq y_0$.

remove can be proved to be a functor.

We want to see if the composition of the two functors equal to identity, and if the two categories are isomorphic. It can be easily proved that $\text{remove} \circ (- \uplus \{\text{err}\}) = id_{\text{Par}}$. However, the opposite does not hold. A counterexample would be $\{0\} \ni 0$. The functor remove sends $\{0\} \ni 0$ to the empty set $\emptyset$ in Par, and if we apply $(- \uplus \{\text{err}\}$ to the empty set, the resulting object is $(\emptyset \uplus \{\text{err}\}) \ni \sigma_2(\text{err})$. Therefore, the two categories are not isomorphic to each other. We need to define a new concept to depict the relationship between them.

2 Equivalence of Sets

Definition 3. A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if

- There exists an inverse functor $F^{-1} : \mathcal{D} \to \mathcal{C}$;
- $F \circ F^{-1} \cong id_{\mathcal{D}}$, i.e., $F \circ F^{-1}$ is a natural isomorphism in $\mathcal{D} \Rightarrow \mathcal{D}$;
- $F^{-1} \circ F \cong id_{\mathcal{C}}$, i.e., $F^{-1} \circ F$ is a natural isomorphism in $\mathcal{C} \Rightarrow \mathcal{C}$.

For the motivating example, it is easy to show that $\text{remove} \circ (- \uplus \{\text{err}\}) \cong id_{\text{Par}}$, since they are already equal. However, it can be tedious to establish $(- \uplus \{\text{err}\}) \circ \text{remove} \cong id_{\text{Set}_*}$, which involves the following steps:

1. For any $X \ni x_0$, define a function from $(X - x_0) \uplus \{\text{err}\} \ni \text{err}$ to $X \ni x_0$;
2. Prove the naturality of the above function
3. For any $X \ni x_0$, define a function from $X \ni x_0$ to $(X - x_0) \uplus \{\text{err}\} \ni \text{err}$;
4. Prove the naturality of the above function.

The above process can be very time-consuming, but fortunately, it can be simplify with the following definitions and theorems.

3 Natural Isomorphism

Theorem 1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be functors between the two categories. Let $\alpha : F \Rightarrow G$ be a natural transformation. α is an isomorphism if and only if $\forall X \in \mathcal{C}_0$, α_X is an isomorphism.*

Proof. The forward direction: suppose $\alpha : F \Rightarrow G$ is an isomorphism. Therefore, there exists $\alpha^{-1} : G \Rightarrow F$ such that $\alpha^{-1} \circ \alpha = id_F$. Hence for any $X \in \mathcal{C}_0$, we have $\alpha_X^{-1} \cdot \alpha_X = id_X$.

The backward direction: suppose for any $X \in \mathcal{C}_0$, α_X is an isomorphism. We use diagrammatic reasoning to show that α is an isomorphism. Since α is a natural tranformation, we have that the right square commutes. By our assumption, α_X and α_Y are both isomorphisms, and therefore, $\alpha_X^{-1} \circ \alpha_X = id_{GX}$ and $\alpha_Y^{-1} \circ \alpha_Y = id_{GY}$, and thus the large rectangle commutes. Hence, the left square commutes, i.e., $\alpha_X^{-1} \circ Ff = Gf \circ \alpha_Y^{-1}$, and thus α_x^{-1} is natural.

$$
\begin{tikzcd}
GX \arrow[r, "\alpha_X^{-1}"] \arrow[d, "Gf"] & FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"] & GX \arrow[d, "Gf"] \\
GY \arrow[r, "\alpha_Y^{-1}"] & FY \arrow[r, "\alpha_Y"] & GY
\end{tikzcd}
$$

□

The following theorem simplifies the proof of equivalence between categories.

Theorem 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and $F : \mathcal{C} \to \mathcal{D}$ be a functor. F is an equivalence of the categories if and only if*

1. *F is* faithful*, i.e., for all $X, Y \in \mathcal{C}$, $F_1^{X,Y} : \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ is injective.*
2. *F is* full [1]*, i.e., for all $X, Y \in \mathcal{C}$, $g : \mathcal{D}(FX, FY)$, there exists $F_1^{-1}g : \mathcal{C}(X, Y)$ such that $F_1(F_1^{-1}g) = g$.*
3. *F is* essentially surjective*, i.e., forall $A \in \mathcal{D}$, there exists $F^{-1}A \in \mathcal{C}$ such that $F(F^{-1}A) \cong A$.*

The following theorem states that functors preserve isomorphism.

Theorem 3. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $F : \mathcal{C} \to \mathcal{D}$ be a functor. If $f : \mathcal{C}(X, Y)$ is an isomorphism, then $Ff : \mathcal{D}(FX, FY)$ is also an isomorphism.*

Proof. Since f is an isomorphism, there exists $f^{-1} : \mathcal{C}(Y)$ such that $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$. By properties of a functor, we have

$$Ff \circ Ff^{-1} = F(f \circ f^{-1}) = F(id_Y) = id_{FY}.$$

Similarly, it can be shown that $Ff^{-1} \circ Ff = id_{FX}$. Therefore, Ff is an isomorphism.

□

[1] A functor is *fully faithful* when it is both faithful and full

Similar properties also hold for split epimorphism and split monomorphism, but not true for monomorphisms and epimorphisms. A counterexample where functor does not preserve epimorphisms, would be the forgetful functor from the category of monoids to the category of sets. The morphism $i : \mathbb{N} \to \mathbb{Z}$ is epi in Monoid but not in Sets.

4 Special Categories

The category Iso has two objects $\{X, Y\}$ and two (non-identity) morphisms f : $\text{Iso}(X, Y)$ and $f^{-1} : \text{Iso}(Y, X)$ satisfying $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$. Given a category $\mathcal{C}$, an isomorphism in $\mathcal{C}$ is equivalent to a functor $i : \text{Iso} \to \mathcal{C}$.

$$\begin{tikzcd} X \arrow[loop left, "id_X"] \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "f^{-1}"] \arrow[loop right, "id_Y"] \end{tikzcd}$$

The category Section has two objects $\{X, Y\}$ and three (non-identity) morphisms $s : \text{Section}(X, Y)$, $r : \text{Section}(Y, X)$ and $s \circ r : \text{Section}(Y, Y)$, satisfying $r \circ s = id_X$. A section in category $\mathcal{C}$ can be represented by a functor from Section to $\mathcal{C}$.

$$\begin{tikzcd} X \arrow[loop left, "id_X"] \arrow[r, bend left, "s"] & Y \arrow[l, bend left, "r"] \arrow[loop right, "id_Y"] \arrow[loop above, "s \circ r"] \end{tikzcd}$$

5 More Examples of Non-trivial Equivalences

5.1 Predicate and Powerset

This is not an example of equivalences of categories, but of preorders. For a given set X, define the preorder of predicates as

- $\text{Pred}(X) := X \to \mathbb{B}$;
- For $P, Q \in \text{Pred}(X)$, $P \leq_{\text{Pred}} Q$ if P implies Q.

The preorder of subsets is defined as

- $\mathcal{P}(X) := \{S \mid S \subseteq X\}$;
- For $S_1, S_2 \in \mathcal{P}(X)$, $S_1 \leq_{\mathcal{P}} S_2$ if $S_1 \subseteq S_2$.

We can define functions of elements between the two preorders :

- $P : X \to \mathbb{B} \mapsto \{x : X \mid P(x) = T\}$
- $S \subseteq X \mapsto - \in S$.

Both are order-preserving.

5.2 Families and Slices

Given a set X, the category $\mathrm{Fam}(X)$ is defined as the discrete category on $X \to \mathrm{Set}$, and can be understood as "X-indexed sets". An object in the category $\mathrm{Fam}(X)$ is denoted by $(Y_x)_{x \in X}$. The category Set/X with objects $(Y : \mathrm{Set}) \times (\pi : Y \to X)$ is the slices of sets. An object in the category Set/X is denoted by $Y \xrightarrow{\pi} X$.

$$\begin{tikzcd} \mathrm{Fam}(X) \arrow[r, bend left, "\Sigma"] & \mathrm{Set}/X \arrow[l, bend left, "(-)^{-1}"] \end{tikzcd}$$

We define functors between the two categories as follows:

- $\Sigma(Y_x)_{x \in X} := \{(x, y) \mid x \in X, y \in Y_x\}$;
- $(Y \xrightarrow{\pi} X)^{-1} := \{y \mid \pi(y) = x\}$.