

Gradual Typing for Effect Handlers

MAX S. NEW, University of Michigan, USA

ERIC GIOVANNINI, University of Michigan, USA

DANIEL R. LICATA, Wesleyan University, USA

We present a gradually typed language, GrEff, with effects and handlers that supports migration from unchecked to checked effect typing. This serves as a simple model of the integration of an effect typing discipline with an existing effectful typed language that does not track fine-grained effect information. Our language supports a simple module system to model the programming model of gradual migration from unchecked to checked effect typing in the style of Typed Racket.

The surface language GrEff is given semantics by elaboration to a core language Core GrEff. We equip Core GrEff with an inequational theory for reasoning about the semantic error ordering and desired program equivalences for programming with effects and handlers. We derive an operational semantics for the language from the equations proveable in the theory. We then show that the theory is sound by constructing an operational logical relations model to prove the graduality theorem. This extends prior work on embedding-projection pair models of gradual typing to handle effect typing and subtyping.

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1 INTRODUCTION

Gradually typed programming languages are designed to support smooth migration from a lax to a strict static type discipline [22, 26]. Most commonly, gradually typed languages add a static type system to an existing dynamically typed language and allow for (1) safe interoperability between the languages and (2) semantic guarantees that adding types to existing programs only results in stricter type enforcement, and no other behavioral change. More generally, gradual typing has been applied to provide a spectrum of precision in other kinds of typing disciplines such as refinement typing or effect typing [1, 9], where the “dynamic” side is a statically typed language itself.

One particular presentation of effects and effect typing that is gaining popularity is **effect handlers** [19]. Operationally, effect handlers can be **resumable exceptions**, code can “raise” an effect operation, which will then be handled by the closest enclosing handler, which in addition to the exception data will also receive the continuation for the raising code that can be invoked to resume at the original point where the effect was raised. Effect handlers provide a simple typed interface to delimited continuations. Many implementations of effect handlers in experimental languages or as libraries [2, 4, 8, 10, 11], and notably have been added as a built-in language feature in OCaml [24] and are a proposed extension to WASM [3].

Designers of languages supporting effect handlers, much like designers of languages with exceptions, are left with a choice of whether the type system should merely track that the input and

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output types of effect operations are satisfied, or if a an *effect typing* system should be employed to determine which effects can be raised at any given time. On the one hand, checked effects allow programmers to easily reason about which effects can be raised and handle them appropriately. On the other hand, strict checking may necessitate large code changes when code is extended to raise new operations, and many statically typed languages such as Standard ML and Java support unchecked exceptions. Gradual typing can provide a framework in which the programmer is not locked in to one system or another: they might use unchecked exceptions in one module and checked exceptions in another, while supporting well-defined interoperability with useful error messages at runtime if there is an un-handled effect.

In this work we present the design and semantics of GrEff, a gradual language with effect handlers that supports gradual migration from unchecked effects to precise effect typing. The untracked sublanguage of GrEff is designed to be similar to SML and Java’s treatment of exceptions: new effect operations are declared with specified input and output types, and these can be imported and used to raise and handle those operations in other modules, but which effects are raised by a function is not tracked by the type system. But GrEff also supports *tracked* function types $A \rightarrow_{\sigma} B$ where the input values must be of type A , output values will be of type B , but also the function may raise any of the effects in the set σ . The untracked function type is modeled then as a type $A \rightarrow_{?} B$ which has a “dynamic” effect type, in the sense that it may raise any effect, possibly including unknown effect operations declared in some independent module of the program.

There are two aspects in designing a sound gradually typed language: designing the syntax and gradual type checking of the surface language and designing the corresponding core language and semantics. We designed the surface language with the goal of modeling program migration from static to dynamic typing. For this reason we include a simple module system in the style of Typed Racket [26] so that we can express that different portions of the program have different views on how the effect operations are typed. Once the base language is designed, the gradual type system is based on prior work on gradual type systems [7, 22].

To design the core language and runtime semantics, we follow the prior work ([14, 17]) which established a recipe for designing a new gradual language to satisfy the graduality theorem and validate strong type-based equational reasoning principles. Their approach is to *axiomatize* the type-based reasoning principles as equations and the graduality theorem as inequalities, where casts are defined not by specifying their operational behavior but instead by assuming they are given by least upper bounds/greatest lower bounds. Then the operational behavior of the casts can be *derived* from the inequational theory. An operational or denotational model must then be constructed to prove the theory is sound, which implies the graduality theorem. But since the operational semantics is *derived* from the inequational theory, this also establishes a stronger theorem that the observable behavior or the casts is *uniquely determined* by the desired type-based reasoning and graduality, showing that any observably different cast semantics must violate one or more of the axioms.

For Core GrEff we extend this recipe to include *effect casts* and *subtyping* of value and effect types. We then show that every rule of an operational semantics is derivable from the least upper bound/greatest lower bound specifications of casts as well as congruence rules and a *handler extensionality* principle for effect handlers that is motivated by the free monad models of effect handlers. The handler extensionality principle states that a handler clause that simply re-raises the effect it handles with the same continuation can be removed without changing the behavior, a desirable optimization in any effect handler system.

For ordinary gradual typing with a dynamic type, the dynamic type is modeled by a recursive sum type of all of the basic type formers. Then all gradual types A come with a specified *embedding projection pair* of casts which say how to embed the type $e : A \rightarrow ?$ and vice-versa, how to cast

(with possibility of errors) from $p : ? \rightarrow A$. The defining properties of an embedding-projection pair are that embedding followed by projecting is the identity $p \circ e = \text{id}$ while projecting and then embedding is *below* the identity in the *error ordering* $e \circ p \sqsubseteq \text{id}$. Then gradual type casts from a type A to a type B are modeled by casting through the dynamic type by first embedding A into $?$ and then projecting to B . While this is not necessarily an efficient implementation, it is a simple specification for how an optimized implementation should extensionally behave. GrEff has, rather than a dynamic *value type*, a dynamic *effect type*, and so we must adapt the notion of universal type and embedding-projection pair to this setting. We designed GrEff’s core language based on a heuristic denotational semantics. Each effect type σ is interpreted as a *free monad* T_σ , a standard semantics for algebraic effects: $T_{\varepsilon_i:A_i \rightsquigarrow B_i} \dots A = \mu X. A + (A_i \times (B_i \rightarrow X)) + \dots$. An element of the free monad is either a final value A or a choice of operation ε_i with a corresponding request value, and a kontinuation $B_i \rightarrow T \dots A$ for the response which is a further free monad computation. The dynamic effect will be interpreted then as a particular free monad, generated from the “most imprecise” version of the effects possible.

1.1 Nominal vs Structural Effects

In GrEff, new effect operations can be declared in each module, similar to new exceptions being declared in ML-style languages. When an effect is declared in a module, it is given an associated *request* and *response* type. For instance, an effect for reading a boolean state would be `get : 1 ~> 2`, the user provides a trivial element of 1 as the request and receives an element of boolean type 2 as the response, while an effect for writing a boolean state would be `set : 2 ~> 1`. This means that GrEff takes a *nominal* approach to effect operations: each effect operation has an associated request and response type that is used to determine when raising or handling the effect is done correctly. However, having a single, global assignment from effect names to request/response types is problematic from the perspective of *gradual* migration from untracked to tracked effects. In a completely nominal form of effect typing, if an effect operation is used in many different modules with imprecise typing, and one module is migrated to use a more precise version of the effect’s request/response type, then we would need to migrate all modules to use the more precise type. Instead, gradual migration should allow for this to be done a single module at a time. To achieve this, in GrEff, we take a *locally nominal* but *globally structural* approach to the typing of effect operations. That is, each module has a single view of what the request and response types are for each effect operation that is in scope, but different modules can associate different types to the same effect operation.

The contributions of the paper are as follows:

- (1) We define a gradually typed language GrEff supporting migration from unchecked to checked effects and handlers.
- (2) We prove this language satisfies the static gradual guarantee and the dynamic gradual guarantee (graduality).
- (3) We give the language a semantics by elaboration into a core language.
- (4) We axiomatize the desired graduality and program equivalence properties of the core language by giving an inequational theory. We then derive from this an operational semantics by orienting certain equations in the theory, showing the operational behavior is uniquely determined by the graduality and extensionality principles.
- (5) We prove type soundness and graduality by constructing a logical relations model, extending prior work on ep pair semantics to effects and subtyping.

2 OVERVIEW OF GREFF

Before discussing the syntax and semantics of GrEff, we provide an informal introduction to its features and how it supports a gradual migration from unchecked to checked effect handlers. As an example, consider the implementation of a simple threading library using effect handlers. We start with a system using unchecked effect types in an ASCII syntax in Figure 1. We split this program across three modules: first, a module `Operations` defines the effects we will be using in our other modules. These are the effects that the threads use: `print` for displaying output so that we can observe the interleaving of threads, `yield`, which yields back control to the scheduler, and most importantly, `fork`, which allows for a thread to spawn new threads. Each effect declaration `effect e : Req ~> Resp` is annotated with two types: the type of *requests* to the ambient handler, and the type of expected *responses* from the ambient handler. For instance, the request type for `print` is a string to be printed, and the response is unit. In a more realistic setting, the response type might be a boolean to say if the printing succeeded, or an unsigned integer to say how many bytes were successfully printed. For `yield`, the request and response are both unit. For `fork`, the response type is again unit and the request type is a thunk `1 -[?]> 1` where the `?` is the type of *effects* the function may raise when called. In this case, `?` indicates the thunk might raise any effect.

Next, module `Scheduler` defines a round-robin scheduler as a handler for the provided effects. For simplicity the implementation relies on some built-in queue implementation, and shallow handlers, a simple extension to our formalism which uses the more complex deep handlers. Finally, we have the `Main` module, which uses the scheduler defined in the `Scheduler` module with a thunk that uses the effects defined in the `Operations` to implement a program that prints a simple message using threads whose output will depend on the scheduler's behavior.

The imprecision of the effect typing in this program means that programmers have to rely on documentation or understanding of the code to understand what effects might be raised when they import a function from another module. With effect typing, this information can be expressed precisely using effect annotations on the functions themselves. For instance, in the declaration of the `fork` operation, the request is a thunk that when launched as a thread itself may raise further effects such as manipulating shared state, `yielding` to other threads, or `forking` additional threads. However with imprecise effect tracking, the scheduler procedure `rr` has the uninformative type `1 -[?]> 1 -[?]> 1` so we cannot specify in the type which operations the scheduler will handle and which it will propagate forward.

GrEff allows as well for the introduction of *precise* effect types to express these choices in the type structure. In figure 2, we show a fully precisely typed version of the same threading program (with implementations, which are unchanged, now elided). This allows us to specify in the `Scheduler` module that the scheduler expects threads that can (1) `print` a string, (2) `yield` to the other threads and (3) `fork` further threads with the same effects. To express this, the scheduler module changes the type to `1 -[fork,print,yield]> 1 -[> str` expressing that the scheduler will be passed a thunk that may `fork` `print` or `yield`, but will itself return a string without raising any effects. Additionally, we can express that *forked* threads should only raise these three effects as well. This is expressed by annotating the `import` statement, which defines `fork` as a recursive¹ effect type whose response type is trivial and whose request type is that of thunks that can raise the three provided effects. This typing will then be used by all occurrence of the `fork` effect, in `raise` or handlers, within this module. The types are also changed in the main module, where the `letters` thunk can be given a type expressing it only `prints` and `yields`, whereas `numbers` thunk only `forks` and `prints`.

¹though recursive effect types are natural here, we do not support them in our core language and leave this extension to future work

```

module Operations where
  effect yield : 1 -> 1
  effect print : str -> 1
  effect fork  : (1 -[?]> 1) ~> 1
module Scheduler where
  import Operations.yield : 1 ~> 1
  import Operations.print : str -> 1
  import Operations.fork  : (1 -[?]> 1) ~> 1
  define sch-loop : Queue (1 -[?]> 1) -[?]> str -[?]> str = lambda q.
    match q with
      empty          -> ()
      dequeue(thunk, q') -> shallow-handle thunk() with
        ret _ -> sch-loop q'
        fork(new,k) -> sch-loop (enqueue (enqueue q new) k)
        yield(_, k) -> sch-loop (enqueue q k)
        print(s, k) -> lambda s'. k(s' ++ s)
  define scheduler : (1 -[?]> 1) -[?]> str = lambda thunk.
    sch-loop (enqueue empty thunk) ""
module Main where
  import Operations.yield : 1 ~> 1
  import Operations.print : str -> 1
  import Operations.fork  : (1 -[?]> 1) ~> 1
  import Scheduler.scheduler : (1 -[?]> 1) -[?]> str
  define letters : 1 -[?]> 1 =
    print("a"); yield(); print("b"); ()
  define numbers : 1 -[?]> 1 =
    print("1"); fork(letters); print("2"); ()
  define main: 1 -[?]> str =
    scheduler(numbers)

```

Fig. 1. GrEff Threading Program with Imprecise Types

These are compatible with the types in scheduler using an effect subtyping that allows functions that use fewer effects to be used in a context that can handle more.

Since GrEff is a *gradual* effect language, a programmer who started with the imprecise program does not need to fully type the entire program before running it. Instead, the programmer can *gradually* migrate from the imprecise style to the more precise style, for example one module at a time. In fact, any of the $2^3 = 8$ combinations of the imprecise versions and precise versions of the three modules presented here will pass the GrEff gradual type-and-effect checker. For instance, we might start with adding precise effect typing to the Operations module to specify the effects that a forked thread can have. Whereas in a non-gradual type system, this would require changing the consumer modules to use the more precise typing, in GrEff, the import statements allow for the uses within the module to continue to use the imprecise typing, and at the module boundary it is checked that the precise components of the declared type for the fork effect match the precise components of the declaration in the defining module. On the other hand, we can keep the Operations module imprecisely typed, and instead add typing to the Scheduler module first. This is again unusual compared to a conventional typed language, we have declared a nominal data type in one module,

```

module Operations where
  effect yield : 1 -> 1
  effect print : str -> 1
  effect fork  : (1 -[fork,print,yield]> 1) ~> 1
module Scheduler where
  import Operations.yield : 1 ~> 1
  import Operations.print : str -> 1
  import Operations.fork  : (1 -[fork,print,yield]> 1) ~> 1
  define sch-loop : Queue (1 -[fork,print,yield]> 1) -[]> str -[]> str = ...
  define scheduler : (1 -[fork,print,yield]> 1) -[]> str = ...
module Main where
  import Operations.yield : 1 ~> 1
  import Operations.print : str -> 1
  import Operations.fork  : (1 -[fork,print,yield]> 1) ~> 1
  import Scheduler.scheduler : (1 -[fork,print,yield]> 1) -[]> str
  define letters : 1 -[print,yield]> 1 =
    print("a"); yield(); print("b"); ()
  define numbers : 1 -[fork,print]> 1 =
    print("1"); fork(letters); print("2"); ()
  define main: str =
    scheduler(numbers)

```

Fig. 2. GrEff Threading Program with Precise Typing

but use it at a different type in a client module. The import statements allow for the gradual migration of the client code without changing the original library.

The module system plays a crucial role in allowing for the programmer to independently choose between migrating the declaration site of the nominal datatype and its uses. If we were in a purely expression-oriented language, then any change to the module declaration, even in a gradual language, would change the typing of the uses of the operation. Here we use the module boundaries in the style of Typed Racket as a way to formally specify different expectations of what the type of the nominal effect operations should be in different portions of the codebase.

3 SURFACE AND CORE GREFF

In this section, we introduce the syntax and typing of GrEff along with its elaboration into a core language, Core GrEff. GrEff includes a module system and nominal effect operations, as well as a gradual type checking algorithm that allows for a mix of dynamic and static effect tracking. Core GrEff, on the other hand, is a simpler expression language with a declarative type system where all gradual type casts (but not subtyping) are explicit in the term. The high-level features of GrEff are elaborated away into core GrEff. Because Core GrEff is simpler, we describe its syntax and typing first, and then describe GrEff and its type-checking/elaboration algorithm.

3.1 Syntax and Typing of Core GrEff

We give an overview of the Core GrEff syntax in Figure 3. Core GrEff expression syntax include typical lambda calculus syntax for variables, let-bindings, functions and booleans. Next, it includes forms for raising an effect operation $\text{raise } \varepsilon(M)$ and handling effect operations $\text{handle } M \{\text{ret } x.N \mid \phi\}$. The handler includes a clause $\text{ret } x.N$ to handle a return value for M as well as clauses for handling

Terms M, N	::=	$x \mid \lambda x.M \mid MM' \mid \text{true} \mid \text{false} \mid \text{if } M\{N\}\{N\}$ $\mid \text{let } x = M \text{ in } N \mid \text{raise } \varepsilon(M) \mid \text{handle } M \{\text{ret } x.N \mid \phi\}$ $\mid \langle B \triangleleft A \rangle M \mid \langle A \triangleleft B \rangle M \mid \langle \tau \triangleleft \sigma \rangle M \mid \langle \sigma \triangleleft \tau \rangle M$
Handler clause ϕ	\in	$\text{Name} \rightarrow_{\text{fin}} \text{Term}$
Value Types A, B, C	::=	$A \rightarrow_{\sigma} B \mid \text{bool}$
Effect Types σ, τ	::=	$? \mid \sigma_c$
Concrete Effect Types σ_c	\in	$\text{Name} \rightarrow_{\text{fin}} \text{ValueType}^2$
Signature Σ	\in	$\text{Name} \rightarrow_{\text{fin}} \text{NonTrackingType}^2$
Non-tracking Types $A?$::=	$A? \rightarrow? A? \mid \text{bool}$
Typing Contexts Γ	::=	$\cdot \mid \Gamma, x : A$
Values V	::=	$x \mid \lambda x : A.M \mid \text{true} \mid \text{false}$ $\mid \langle A \rightarrow_{\sigma} B \triangleleft A' \rightarrow_{\sigma'} B' \rangle V \mid \langle A' \rightarrow_{\sigma'} B' \triangleleft A \rightarrow_{\sigma} B \rangle V$
Evaluation Context E	::=	$\bullet \mid \langle B \triangleleft A \rangle E \mid \langle A \triangleleft B \rangle E \mid \langle \tau \triangleleft \sigma \rangle E \mid \langle \sigma \triangleleft \tau \rangle E$ $\mid \text{raise } \varepsilon(E) \mid \text{handle } E \{\text{ret } x.N \mid \phi\} \mid EM \mid VE$ $\mid \text{if } E\{N_t\}\{N_f\} \mid \text{let } x = E \text{ in } N$

Fig. 3. Core GrEff Syntax

effects ϕ . Abstracting from syntactic details, ϕ is modeled as a finitely supported partial function from effect names to terms, which all have two free variables x and k for the payload of the effect raised and its continuation. That is, if syntactically a handler has a clause $\varepsilon(x, k) \mapsto N_{\varepsilon}$, we model this by having $\phi(\varepsilon) = N_{\varepsilon}$. Finally, Core GrEff includes four explicit gradual type cast forms: downcasts ($\langle A \triangleleft B \rangle M$) and upcasts ($\langle B \triangleleft A \rangle M$) for value types, as well as analogous casts for effect types ($\langle \sigma \triangleleft \tau \rangle M$ and $\langle \tau \triangleleft \sigma \rangle M$).

The value types A, B, C classify runtime values: in this simple calculus, just booleans and functions, where functions are typed with respect to a domain, codomain as well as an effect type σ which classifies what effects the function may raise when it is called. The effect types are either $?$ to indicate dynamically tracked effects, or a concrete effect type. A concrete effect type says which effect names ε can be raised, and when they are raised, what is the type of the request A the raising party provides and what is the type of responses B with which the handling party can resume. Abstracting from syntactic details, this is defined to be a finitely supported partial mapping from names to pairs of value types. To model that an effect ε can be raised with request type A and response type B we would define $\sigma_c(\varepsilon) = (A, B)$, which we will notate more suggestively as $\varepsilon : A \rightsquigarrow B \in \sigma_c$. As shown in Section 2, programs declare which effect names can be used, and with which associated request and response types. To track this information in typing core GrEff expressions, we type check all GrEff expressions against a *Signature* Σ which associates a pair of *non-tracking* types to each name. By a *non-tracking* type, we mean a value types that only use $?$ effect types. Additionally, expressions are type-checked with respect to an ordinary typing context Γ . Finally, we define typical notions of value and evaluation context to encode a call-by-value, left-to-right evaluation order. Most notably, all casts are evaluation contexts, and function casts are values, i.e. “proxies” that delay type enforcement until an application is performed.

Next, we present *declarative* term typing rules in Figure 4. The main judgment $\Sigma \mid \Gamma \vdash_{\sigma} M : A$ says that under the assumptions Γ , M can raise effects drawn from σ , and produce a final value of type A . We follow the convention that whenever we form the judgment $\Sigma \mid \Gamma \vdash_{\sigma} M : A$ we must already have established that the types in Γ, A, σ are well-formed under the signature Σ . First, we include explicit subsumption rules for value and effect subtyping, which we will soon define. The rules for value forms (variable, booleans, and lambdas) all have an arbitrary effect type σ because

$$\begin{array}{c}
\frac{\Sigma \mid \Gamma \vdash_{\sigma} M : A \quad \Sigma \mid \Gamma \vdash A \leq : B}{\Sigma \mid \Gamma \vdash_{\sigma} M : B} \quad \frac{\Sigma \mid \Gamma \vdash_{\sigma} M : A \quad \Sigma \mid \Gamma \vdash \sigma \leq : \tau}{\Sigma \mid \Gamma \vdash_{\tau} M : A} \quad \frac{\Gamma(x) = A}{\Sigma \mid \Gamma \vdash_{\sigma} x : A} \\
\\
\Sigma \mid \Gamma \vdash_{\sigma} \text{true, false} : \text{bool} \quad \frac{\Sigma \mid \Gamma, x : A \vdash_{\tau} M : B}{\Sigma \mid \Gamma \vdash_{\sigma} \lambda x. M : A \rightarrow_{\tau} B} \quad \frac{\Sigma \mid \Gamma \vdash_{\sigma} M : A \quad \Sigma \mid \Gamma, x : A \vdash_{\sigma} N : B}{\Sigma \mid \Gamma \vdash_{\sigma} \text{let } x = M \text{ in } N : B} \\
\\
\frac{\Sigma \mid \Gamma \vdash_{\sigma} M : A \rightarrow_{\sigma} B \quad \Sigma \mid \Gamma \vdash_{\sigma} N : A}{\Sigma \mid \Gamma \vdash_{\sigma} MN : B} \quad \frac{\Sigma \mid \Gamma \vdash_{\sigma} M : \text{bool} \quad \Sigma \mid \Gamma \vdash_{\sigma} N_t : B \quad \Sigma \mid \Gamma \vdash_{\sigma} N_f : B}{\Sigma \mid \Gamma \vdash_{\sigma} \text{if } M \{N_t\}\{N_f\} : B} \quad \frac{\Sigma \mid \Gamma \vdash_{\sigma} M : A \quad \epsilon @ A \rightsquigarrow B \in \sigma}{\Sigma \mid \Gamma \vdash_{\sigma} \text{raise } (\epsilon @ A \rightsquigarrow B)(M) : B} \\
\\
\frac{\Sigma \mid \Gamma \vdash_{\sigma} M : A \quad \Sigma \mid \Gamma, x : A \vdash_{\tau} N : B \quad (\forall (\epsilon : A_{\epsilon} \rightsquigarrow B_{\epsilon}) \in \sigma. (\epsilon \notin \text{dom}(\phi) \wedge (\epsilon : A_{\epsilon} \rightsquigarrow B_{\epsilon}) \in \tau) \vee (\Sigma \mid \Gamma, x : A_{\epsilon}, k : B_{\epsilon} \rightarrow_{\tau} B \vdash_{\tau} \phi(\epsilon) : B))}{\Sigma \mid \Gamma \vdash_{\tau} \text{handle } M \{ \text{ret } x.N \mid \phi \} : B} \quad \frac{\Sigma \mid \Gamma \vdash_{\sigma} M : A \quad \Sigma \vdash A \sqsubseteq B}{\Sigma \mid \Gamma \vdash_{\sigma} \langle B \prec_{\tau} A \rangle M : B} \\
\\
\frac{\Sigma \mid \Gamma \vdash_{\sigma} M : B \quad \Sigma \vdash A \sqsubseteq B}{\Sigma \mid \Gamma \vdash_{\sigma} \langle A \leftarrow B \rangle M : A} \quad \frac{\Sigma \mid \Gamma \vdash_{\sigma} M : A \quad \Sigma \vdash \sigma \sqsubseteq \sigma'}{\Sigma \mid \Gamma \vdash_{\sigma'} \langle \sigma' \prec_{\tau} \sigma \rangle M : A} \quad \frac{\Sigma \mid \Gamma \vdash_{\sigma'} M : A \quad \Sigma \vdash \sigma \sqsubseteq \sigma'}{\Sigma \mid \Gamma \vdash_{\sigma} \langle \sigma \leftarrow \sigma' \rangle M : A}
\end{array}$$

Fig. 4. Core Greff Typing

they do not raise any effects themselves. The let, application and if rules simply require that all the sub-terms use the same effect type, though subsumption can be used to combine effects. The raise rule says that the effect being raised needs to be in the current effect type and the payload of the request must also have the same effect type.

Next, the rule for typing a handler works as follows. First, we have the output value is B and output effect types is τ , while for the scrutinee M the corresponding types are A and σ . First, we check that the return clause N has the same output types as the handler overall, when its input x has the type of the output of M . Next, for each effect operation $\epsilon : A_{\epsilon} \rightsquigarrow B_{\epsilon}$ raised by M , either the effect is not handled by ϕ , in which case it must be included in the final effect type, or it is handled by ϕ . If it is handled by ϕ , then the clause $\phi(\epsilon)$ must be well typed with a request value $x : A_{\epsilon}$ and a continuation that takes responses and has output effect and value types that match the term overall $k : B_{\epsilon} \rightarrow_{\tau} B$. Lastly, we include the rules for type and effect upcasts and downcasts. Whenever a *type precision* relationship $A \sqsubseteq B$ holds (to be defined), we get an *upcast* from the more precise type A to the more imprecise type B and a corresponding downcast from B to A .

Finally, finishing out the syntax, in Figure 5, we define three judgments on types: well-formedness, subtyping and type precision. Well-formedness $\Sigma \vdash A$ and $\Sigma \vdash \sigma$ checks that the types used in effect operations erase to the types associated in the signature. Here we use the notation $|A|$ to mean the erasure of effect typing information in that we replace any effect type subterms σ with dynamic $?$. Subtyping works as usual for booleans and functions, contravariant in domain of the function type, but covariant in the codomain and effect. Subtyping for effect types includes both a *width* subtyping aspect: a smaller type can raise fewer operations, as well as a *depth* aspect that is *covariant* in the request type and *contravariant* in the response type. This variance sense from the perspective of the party *producing* the request, to match the function type subtyping. Finally,

$$\begin{array}{c}
\Sigma \vdash \text{bool} \quad \frac{\Sigma \vdash A \quad \Sigma \vdash \sigma \quad \Sigma \vdash B}{\Sigma \vdash A \rightarrow_{\sigma} B} \quad \Sigma \vdash ? \quad \frac{\forall \varepsilon : A \rightsquigarrow B \in \sigma_c. \quad (\varepsilon : |A| \rightsquigarrow |B| \in \Sigma). \wedge \Sigma \vdash A \wedge \Sigma \vdash B}{\Sigma \vdash \sigma_c} \\
\\
\text{bool} \leq: \text{bool} \quad \frac{A' \leq: A \quad \sigma \leq: \sigma' \quad B \leq: B'}{A \rightarrow_{\sigma} B \leq: A' \rightarrow_{\sigma'} B'} \quad ? \leq: ? \quad \frac{\forall \varepsilon : A_{\sigma} \rightsquigarrow B_{\sigma} \in \sigma_c. \exists \varepsilon : A_{\tau} \rightsquigarrow B_{\tau} \in \tau_c. \quad A_{\sigma} \leq: A_{\tau} \wedge B_{\tau} \leq: A_{\tau}}{\sigma_c \leq: \tau_c} \\
\\
\text{bool} \sqsubseteq \text{bool} \quad \frac{A \sqsubseteq A' \quad \sigma \sqsubseteq \sigma' \quad B \sqsubseteq B'}{A \rightarrow_{\sigma} B \sqsubseteq A' \rightarrow_{\sigma'} B'} \quad \sigma \sqsubseteq ? \quad \frac{\text{dom}(\sigma_c) = \text{dom}(\sigma'_c) \quad \forall \varepsilon : A \rightsquigarrow B \in \sigma_c. \exists \varepsilon : A' \rightsquigarrow B' \in \sigma'_c. \quad A \sqsubseteq A' \wedge B \sqsubseteq B'}{\sigma_c \sqsubseteq \sigma'_c}
\end{array}$$

Fig. 5. Well formed types and effects, Type and Effect Precision

Programs P	$::=$	$L; \dots L_{\text{main}}$
Value Types A, B, C	$::=$	$A \rightarrow_{\sigma} B \mid \text{bool}$
Effect Types σ, τ	$::=$	$? \mid \sigma_s$
Operation Set σ_s, τ_s	\in	$\mathcal{P}_{\text{fin}}(\text{Name})$
Values V	$::=$	$x \mid \lambda x : A. M \mid \text{true} \mid \text{false}$
Terms M, N	$::=$	$x \mid \text{raise } \varepsilon(M) \mid \text{handle}_{C! \sigma} M \{ \text{ret } x.N \mid \phi \}$ $\mid \lambda x. M \mid M M' \mid \text{true} \mid \text{false} \mid \text{if } M \{ N \} \{ N \}$ $\mid M :: A \mid M :: \sigma$
Handler clauses ϕ	\in	$\text{Name} \rightarrow_{\text{fin}} \text{Term}$
Modules L	$::=$	$\text{module } m \{ b \}$
Module Body b	$::=$	$\cdot \mid D; b$
Main Module L_{main}	$::=$	$\text{main} \{ b; M \}$
Module reference r	$::=$	$m.x \mid m.\varepsilon$
Declaration D	$::=$	$\text{import-eff } r \text{ as } \varepsilon @ A \rightsquigarrow B \mid \text{effect } \varepsilon : A \rightsquigarrow B$ $\mid \text{define } x = V \mid \text{import-val } r \text{ as } x @ A$
Program Typing Contexts Δ	$::=$	$\cdot \mid \Delta, x \mapsto \Gamma_s$
Module Typing Contexts Γ_s	$::=$	$\cdot \mid \Gamma_s, \varepsilon : A \rightsquigarrow B \mid \Gamma_s, x : A$

Fig. 6. GrEff Syntax

type precision $A \sqsubseteq B$ tracks instead how “dynamic” or “imprecise” a type is. For functions it is covariant in every argument, and for effect types, the dynamic effect is the most imprecise and for two concrete effect sets, it has a depth rule that that is covariant in request and response positions.

3.2 Syntax and Elaboration of GrEff

We present the syntax for the surface language GrEff in Figure 6. A GrEff program P consists of a sequence of modules ending in a single “main” module. Each module m consists of two parts: first, the effect definitions and then the value definitions, whose types annotations may use the effects previously defined in that module. An effect definition is either a declaration of a new effect operation effect $\varepsilon : A \rightsquigarrow B$ or an import of an existing effect operation $\text{import-eff } m \text{ as } \varepsilon @ A \rightsquigarrow B$.

$$\begin{array}{c}
\frac{\Sigma \mid \Delta \mid \cdot \vdash b \Rightarrow \Sigma'; \gamma; \Gamma \quad \Gamma \vdash M_s \Rightarrow M : \sigma ! A}{\Sigma \mid \Delta \vdash \text{main } b M_s \Rightarrow \Sigma' \vdash_{\sigma} \text{let } \Gamma = \gamma \text{ in } M : A} \quad \frac{\Sigma \mid \Delta \mid \cdot \vdash b \Rightarrow \Sigma'; \gamma; \Gamma \quad \Sigma, \Sigma' \mid \Delta, m \mapsto \Gamma \vdash P \Rightarrow \Sigma'' \vdash_{\sigma} M : A}{\Sigma \mid \Delta \vdash \text{module } m b; P \Rightarrow \Sigma', \Sigma'' \vdash_{\sigma} \text{let } \Gamma = \gamma \text{ in } M : A} \\
\\
\Sigma \mid \Delta \mid \Gamma \vdash \cdot \Rightarrow \cdot; \cdot; \cdot \quad \frac{\varepsilon \notin \Sigma \quad \Gamma \vdash A_s \Rightarrow A \quad \Gamma \vdash B_s \Rightarrow B}{\Sigma \mid \Delta \mid \Gamma \vdash \text{effect } \varepsilon : A_s \rightsquigarrow B_s \Rightarrow (\varepsilon @ [A] \rightsquigarrow [B]); \cdot; \varepsilon @ A \rightsquigarrow B} \\
\\
\frac{\Sigma \mid \Delta \mid \Gamma \vdash D \Rightarrow \Sigma'; \gamma'; \Gamma' \quad \Sigma, \Sigma' \mid \Delta \mid \Gamma, \Gamma' \vdash b \Rightarrow \Sigma''; \gamma''; \Gamma''}{\Sigma \mid \Delta \mid \Gamma \vdash D; b \Rightarrow \Sigma', \Sigma''; \gamma', \gamma''; \Gamma', \Gamma''} \quad \frac{\Delta(m) \ni \varepsilon @ A' \rightsquigarrow B' \quad \Gamma \vdash A_s \Rightarrow A \quad \Gamma \vdash B_s \Rightarrow B \quad A \sim A' \quad B \sim B'}{\Sigma \mid \Delta \mid \Gamma \vdash \text{import-eff } m \text{ as } \varepsilon @ A_s \rightsquigarrow B_s \Rightarrow \cdot; \cdot; \varepsilon @ A \rightsquigarrow B} \\
\\
\frac{\Gamma \vdash V_s \Rightarrow V : \emptyset ! A}{\Sigma \mid \Delta \mid \Gamma \vdash \text{define } x = V_s \Rightarrow \cdot; V/x; x : A} \quad \frac{\Delta(m) \ni x : A' \quad \Gamma \vdash A_s \Rightarrow A \quad A' \lesssim A}{\Sigma \mid \Delta \mid \Gamma \vdash \text{import-val } m.x \text{ as } y @ A_s \Rightarrow \cdot; \langle A \leftarrow A' \rangle x/y; y : A}
\end{array}$$

Fig. 7. GrEff Typing/Elaboration, Module Language

In either case, the declaration includes the request type A and the response type B of the effect. An effect import brings an effect defined in another module into the current scope, but with a possibly different request and response type. To support *gradual* migration, these types are allowed to have a different level of precision than the original, but where both are precise they must match. After the effect declarations are the value definitions which are also either a definition of a new value $\text{define } x = V$ or an import of a value declared in a different module at a possibly different type $\text{import-val } r \text{ as } x @ A$. For simplicity, all effects and values are public and can be imported by later modules. Finally a program ends with a main module, which consists of the same kind of effect and value declarations, followed by a final main expression.

Next, we present the elaborator from GrEff into core GrEff, which also serves as the type checker. We view GrEff programs as essentially a description of an effect signature Σ and a closed expression well-typed under that signature. The module system is a way to manage the declaration of new effect operations in the signature and a way to manage the typing of effect operations by giving nominal associations to request and response types rather than solely the structural typing in core GrEff. We describe the elaboration of the module language in Figure 7. The top-level judgment $\Sigma \mid \Delta \vdash P \Rightarrow \Sigma' \vdash_{\sigma} M : A$ says that under the starting signature Σ and previously defined modules Δ , we can elaborate P to a term M with effect type σ and value type A that is well-typed under the extension of the signature by Σ' . This expresses that not only does a program denote a core GrEff program, but it also has a “side effect” of allocating new effect names Σ' . A module is elaborated with the judgment $\Sigma \mid \Delta \mid \Gamma \vdash b \Rightarrow \Sigma'; \gamma'; \Gamma'$. The outputs of this judgment are the newly allocated effects of the module Σ' , the names of effect operations and types for values the module defines Γ' and the definitions of all the values the module defines, given as a substitution γ' from names in Γ' to terms of their associated types. Then to elaborate a program consisting of several modules, first you elaborate the modules and then elaborate the remainder of the program and finally combine the two by let-binding all of the names declared in the module, which we write as a shorthand $\text{let } \Gamma = \gamma \text{ in } M$. Note that though Γ contains both variables and effect declarations, the effect declarations are unused in this part of the elaboration. A module is elaborated by combining the results of elaborating each declaration. A new effect declaration checks that the name is not previously declared, and then recursively elaborates the syntactic types declared for request and response and then adds these to the allocated effects as well as the local effect names declared in the module. When adding to the signature, we take the “ceiling” of the types because signatures use

$$\begin{array}{c}
\langle A \Leftarrow B \rangle M = \langle A \Leftarrow [A] \rangle \langle [B] \Leftarrow B \rangle M \qquad \langle \sigma \Leftarrow \tau \rangle M = \langle \sigma \Leftarrow ? \rangle \langle ? \Leftarrow \tau \rangle M \\
\\
\frac{\Gamma \ni x : A}{\Gamma \vdash x \Rightarrow x : \emptyset ! A} \qquad \Gamma \vdash \text{true} \Rightarrow \text{true} : \emptyset ! \text{bool} \qquad \Gamma \vdash \text{false} \Rightarrow \text{false} : \emptyset ! \text{bool} \\
\\
\frac{\Gamma \vdash M_s \Rightarrow M : \sigma ! A' \quad \Gamma \vdash A_s \Rightarrow A \quad A' \lesssim A}{\Gamma \vdash M_s :: A_s \Rightarrow \langle A \Leftarrow A' \rangle M : \sigma ! A} \qquad \frac{\Gamma \vdash M_s \Rightarrow M : \sigma' ! A \quad \Gamma \vdash \sigma_s \Rightarrow \sigma \quad \sigma' \lesssim \sigma}{\Gamma \vdash M_s @ \sigma_s \Rightarrow \langle \sigma \Leftarrow \sigma' \rangle M : \sigma ! A} \\
\\
\frac{\Gamma \vdash M_s \Rightarrow M : \sigma_m ! \text{bool} \quad \Gamma \vdash N_s \Rightarrow N : \sigma_n ! B \quad \Gamma \vdash N'_s \Rightarrow N' : \sigma'_n ! B' \quad C = B \tilde{\vee} B' \quad \sigma = \sigma_m \tilde{\vee} \sigma_n \tilde{\vee} \sigma'_n}{\Gamma \vdash \text{if } M_s \{N_s\} \{N'_s\} \Rightarrow \text{if } \langle \sigma \Leftarrow \sigma_m \rangle M \{ \langle \sigma \Leftarrow \sigma_n \rangle \langle C \Leftarrow B \rangle N \} \{ \langle \sigma \Leftarrow \sigma'_n \rangle \langle C \Leftarrow B' \rangle N' \} : \sigma ! \text{bool}} \\
\\
\frac{\Gamma \vdash A_s \Rightarrow A \quad \Gamma, x : A \vdash M_s \Rightarrow M : \sigma ! B \quad \Gamma \vdash M_s \Rightarrow M : \sigma_m ! A \rightarrow_{\sigma_o} B \quad \Gamma \vdash N_s \Rightarrow N : \sigma_n ! A' \quad A' \lesssim A \quad \sigma = \sigma_m \tilde{\vee} \sigma_n \tilde{\vee} \sigma_o}{\Gamma \vdash \lambda x : A_s. M_s \Rightarrow \lambda x. M : \emptyset ! A \rightarrow_{\sigma} B \quad \Gamma \vdash M_s N_s \Rightarrow \langle \langle \sigma \Leftarrow \sigma_m \rangle M \rangle \langle \langle A \Leftarrow A' \rangle \langle \sigma \Leftarrow \sigma_n \rangle N \rangle : \sigma ! B} \\
\\
\frac{\Gamma \vdash M_s \Rightarrow M : \sigma_m ! A' \quad \Gamma \ni \varepsilon @ A \rightsquigarrow B \quad A' \lesssim A \quad \sigma = \sigma_m \tilde{\vee} \{ \varepsilon \}}{\Gamma \vdash \text{raise } \varepsilon (M_s) \Rightarrow \text{let } x = \langle \sigma \Leftarrow \sigma_m \rangle M \text{ in } \langle \sigma \Leftarrow \{ \varepsilon @ A \rightsquigarrow B \} \rangle \text{raise } \varepsilon \langle \langle A \Leftarrow A' \rangle x \rangle : \sigma ! B} \\
\\
\frac{\Gamma \vdash C_s \Rightarrow C \quad \Gamma \vdash \sigma_s \Rightarrow \sigma \quad \Gamma \vdash M_s \Rightarrow M : \sigma_m ! A_m \quad \Gamma, x : A \vdash N_s \Rightarrow N : \sigma_n ! C_n \quad \sigma_n \lesssim \sigma \quad C_n \lesssim C \quad \text{dom}(\phi_{\Leftarrow}) = \text{dom}(\phi_s) \quad (\forall \varepsilon \in \text{dom}(\phi_s). \exists (\varepsilon @ A_\varepsilon \rightsquigarrow B_\varepsilon) \in \Gamma. \Gamma, x : A_\varepsilon, k : B_\varepsilon \rightarrow_{\sigma} C \vdash \phi_s(\varepsilon) \Rightarrow N_\varepsilon : \sigma_\varepsilon ! C_\varepsilon \quad \sigma_\varepsilon \lesssim \sigma \wedge C_\varepsilon \lesssim C \wedge \phi_{\Leftarrow} = \langle \sigma \Leftarrow \sigma_\varepsilon \rangle \langle C \Leftarrow C_\varepsilon \rangle N_\varepsilon)}{\Gamma \vdash \text{handleTy}(\sigma_m, \sigma, \text{dom}(\phi_s)) = \sigma'_m} \\
\\
\frac{}{\Gamma \vdash \text{handle}_{\sigma_s ! C_s} M_s \{ \text{ret } x. N_s \mid \phi_s \} \Rightarrow \text{handle } \langle \sigma'_m \Leftarrow \sigma_m \rangle M \{ \text{ret } x. \langle \sigma \Leftarrow \sigma_n \rangle \langle C \Leftarrow C_n \rangle N \mid \phi_{\Leftarrow} \} : \sigma ! C} \\
\\
\frac{\text{dom}(\sigma_c) \subseteq \text{dom}(\tau_c) \cup \sigma_s}{\Gamma \vdash \text{handleTy}(\sigma_c, \tau_c, \sigma_s) = \tau_c \cup \Gamma(\sigma_s)} \qquad \Gamma \vdash \text{handleTy}(?, \tau_c, \sigma_s) = \tau_c \cup \Gamma(\sigma_s) \\
\\
\Gamma \vdash \text{handleTy}(\sigma_c, ?, \sigma_s) = \sigma_c |_{\sigma_s} \uplus [\Gamma(\text{dom}(\sigma_c) - \sigma_s)] \qquad \Gamma \vdash \text{handleTy}(?, ?, \sigma_s) = ?
\end{array}$$

Fig. 8. GrEff Typing/Elaboration, Expression Language

untracked types. Next, to import an effect from a different module, the types given for the effect are checked to be compatible with the types declared in the other module. Here the compatibility judgment $A \sim A'$ is defined as the conjunction of gradual subtyping in both directions, $A \lesssim A'$ and $A' \lesssim A$, to be defined soon. This ensures that any imports from that module using this effect name will succeed. This effect name is added to the local names only, and not the signature, because it is using an already allocated effect name. Next, defining a value simply elaborates the value and adds its type to the output typing and associates the value to that name. Importing a value is similar, except that we check that the declared type is a gradual subtype, and so can be coerced by the cast $\langle A \Leftarrow A' \rangle$, whose definition will be described shortly.

Next, we define the elaboration of the expression language in Figure 8. The judgment $\Gamma \vdash M_s \Rightarrow M : \sigma ! A$ says that under the typing of names given by Γ , the GrEff expression M_s elaborates to the core GrEff function M , which will be well-typed with inferred effect type σ and value type A . All forms essentially elaborate to similar forms in core GrEff, but with suitable casts inserted. First, we define the translation of value type casts $\langle A \Leftarrow B \rangle M$ and effect type casts $\langle \sigma \Leftarrow \tau \rangle M$ as an upcast followed by a downcast. For the effect cast, these casts go through the dynamic effect type, but for two value types there is no single most dynamic effect type so we again use the ceiling operation. Note that this will only be well-typed in case $\lceil B \rceil = \lceil A \rceil$, which is ensured whenever $A \lesssim B$, which is a precondition for inserting a cast. This is not necessarily the most efficient implementation of the cast, we discuss optimizations in Section 4.3

Next, Variables, boolean values and function values elaborate to themselves with an empty effect type \emptyset . The let-binding form shows how different effect types are combined: the effect types of M and N are combined using a gradual join $\tilde{\vee}$, defined in the appendix, and casts are inserted into M and N to give them this effect type. The ascription forms simply check that the appropriate kind of type satisfies a gradual subtyping judgment and inserts a cast. This uses the elaboration of types $\Gamma \vdash A_s \Rightarrow A$, defined below. The if rule checks that the condition has boolean type and gives the output value type as the gradual join of the branches, and the output effect type as the gradual join with the condition expression as well. The application rule is similar except that argument is casted to have the type of the domain of the function and the effect type of the function is joined with the effect types of the terms. Next, we have the raise form, which elaborates to a raise but first let-binds the request term and casts the raise term to have an effect type that is the join of the request term's effect type and the operation's type. This let-binding is useful here so that we can cast the single operation separately from M , which doesn't need a cast because in this case its effect type will always be a subtype of σ . Finally, we have the most complex case, the handle form. The handle form elaborates to a handle form in the core language with casts inserted in each case to make them agree with the ascribed value type C and effect type σ . The request variables and input to the continuations are given by looking up the effect in Γ , while the output is given by the ascription. The most complex part of this elaboration is the cast needed for the scrutinee M . In the core language, we need that all of the effects that M raises but are *not* caught by the handle are in the output type σ . But when σ is dynamic and M has concrete effect type or vice-versa, this is not necessarily true, so in these cases a cast must be inserted that effectively handles all of the "other" effects. This definition is given below in a special elaboration of handle scrutinees sub-case $\Gamma \vdash \text{handleTy}(\sigma, \tau, \sigma_s) = \sigma_o$ where the type σ is the elaborated type of the scrutinee, τ is the elaborated type of the result of the handle expression, and σ_s is the set of effects caught by the handler, where we write $\Gamma(\sigma_s)$ for the map that looks up the currently associated types for each operation in σ_s . If σ and τ are both precise collections of effects, then we check that all of the effects it raises are either caught or still occur in the output type, and we insert a subtyping cast. If σ and σ' are both imprecise? then we will use ? as the result type. If the type of the scrutinee is precise, but the desired output type is dynamic, then we need to upcast all of the unhandled effect operations to their dynamic versions. If the type of the scrutinee is dynamic but the output type is precise, then we *downcast* to include only the union of the output effects and the caught effects, otherwise erroring.

Finally, Figure 9 describes the elaboration of types and gradual subtyping. Value and effect type elaboration $\Gamma \vdash A \Rightarrow B$ is mostly structural except that the rule for concrete effect sets resolves the request and response types of the effect operation based on the context Γ . Next, we describe the mostly standard *gradual* subtyping of value types $A \lesssim B$ and effect types $\sigma \lesssim \tau$ to determine when a dynamic cast $\langle B \Leftarrow A \rangle$ or $\langle \tau \Leftarrow \sigma \rangle$ would reduce to subtyping on the precise portions of the

$$\begin{array}{c}
\Gamma \vdash \text{bool} \Rightarrow \text{bool} \quad \frac{\Gamma \vdash A_s \Rightarrow A \quad \Gamma \vdash B_s \Rightarrow B}{\Gamma \vdash \sigma_s \Rightarrow \sigma} \quad \Gamma \vdash ? \Rightarrow ? \quad \frac{\text{dom}(\sigma_c) = \sigma_s \quad \forall \varepsilon \in \sigma_c. \sigma_c(\varepsilon) = \Gamma(\varepsilon)}{\Gamma \vdash \sigma_s \Rightarrow \sigma_c} \\
\Gamma \vdash A_s \rightarrow_{\sigma_s} B_s \Rightarrow A \rightarrow_{\sigma} B
\end{array}$$

$$\text{bool} \lesssim \text{bool} \quad \frac{A' \lesssim A' \quad \sigma \lesssim \sigma' \quad B \lesssim B'}{A \rightarrow_{\sigma} B \lesssim A' \rightarrow_{\sigma'} B'} \quad ? \lesssim \sigma \quad \sigma \lesssim ? \quad \frac{\forall \varepsilon : A_{\sigma} \rightsquigarrow B_{\sigma} \in \sigma_c. \exists \varepsilon : A_{\tau} \rightsquigarrow B_{\tau} \in \tau_c. \quad A_{\sigma} \lesssim A_{\tau} \wedge B_{\tau} \lesssim A_{\tau}}{\sigma_c \lesssim \tau_c}$$

Fig. 9. Gradual Subtyping and Type Elaboration

types. Note that we define gradual subtyping of types in the core language i.e., after elaboration, so that we can compare effect types across module boundaries that use different typings for the effect names. With this intuition, the definition is like that of subtyping, except that the dynamic effect type is a gradual subtype and supertype than all other effect types. We conclude by noting the following syntactic properties of elaboration, which follow by structural induction.

LEMMA 3.1. *If $\cdot \mid P \vdash \Sigma \Rightarrow M \vdash_A \sigma$: and $\cdot \mid P \vdash \Sigma' \Rightarrow M' \vdash_{A'} \sigma'$: then $\Sigma = \Sigma'$ and $M = M'$ and $\sigma = \sigma'$ and $A = A'$.*

LEMMA 3.2. *Well-formedness of Elaborated Programs If $\cdot \mid \cdot \vdash P \Rightarrow \Sigma \vdash_{\sigma} M : A$, then $\Sigma \mid \cdot \vdash_{\sigma} M : A$.*

4 AXIOMATIC AND OPERATIONAL SEMANTICS

Next we turn to the semantic aspects of GrEff: how expressions are evaluated, what simplifications/optimizations are correct to perform, and that the graduality principle holds for the language. We formalize these three aspects by defining an axiomatic semantics in the form of an *inequational theory* for reasoning about Core GrEff programs. That is, we define a notion of inequality $M \sqsubseteq N$ between expressions called *term precision*, which is a kind of extension of the notion of type precision to expressions. The semantic interpretation of this inequality is that M has the same behavior as N with respect to output and termination, except in that it may raise a dynamic type error when N does not. From this notion inequality we get an induced equivalence relation $M \equiv N$ that specifies when M and N have the same behavior. Term precision and the induced equivalence are used to model our desired semantic ideas: an expression M can be evaluated to a value V when the equivalence $M \equiv V$ holds, M can be simplified/optimized to N when $M \equiv N$ holds, and the graduality principle states that when M is rewritten in the surface language to some M' that has more precise typing information, than a corresponding relationship $M' \sqsubseteq M$ should hold: adding more precise type information results in more precise dynamic type checking. With this in mind, we axiomatize the valid optimizations known from effect handlers as well as desired inequalities from prior work on graduality in our inequational theory.

Axioms are only useful if we can construct models in which they are satisfied. For GrEff, we do this by constructing an *operational* semantics that specifies more precisely how to evaluate programs and then define notions of observational equivalence and an error ordering to model \equiv and \sqsubseteq and prove that all of the axioms are valid in this operational model. We will construct this operational semantics, based on the axiomatics: we show in Section 4.2 that every reduction $M \mapsto N$ is justified by a provable equivalence $M \equiv N$ in the inequational theory. For many rules this is very straightforward, e.g., β reduction of functions is justified by a corresponding β equation. The most utility we get from the axioms in this case is for the cast reductions: cast reductions for

handlers are justified not by a direct corresponding rule in the axioms, but instead by extensionality (η) principles for handlers combined with a least upper bound/greatest lower bound property of casts identified in prior work as being key to the graduality property [16]. This shows that the operational behavior we define has a canonical status: if certain optimizations for handlers are to be valid, and the graduality property is desired, then the cast reductions we define must be used.

4.1 Axiomatic Semantics

We present a selection of the rules of the inequational theory of term precision in Figure 10. The full rules are provided in the appendix. The form of the inequality judgment is $\Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \tau} M \sqsubseteq N : A \sqsubseteq B$, which says that M is more precise, or, roughly, “errors more” than N . This is a kind of *heterogeneous* inequality relation in that M and N are not required to have the same type: M must have value type A and effect type σ and N must have value type B and effect type τ under the context Γ^{\sqsubseteq} and $A \sqsubseteq B$ and $\sigma \sqsubseteq \tau$ must hold. We allow for M and N to be open terms, typed with respect to the typing context Γ^{\sqsubseteq} . The typing context Γ^{\sqsubseteq} is like an ordinary typing context Γ , except that variables are typed $x : A \sqsubseteq B$ where the left type A is the type x has in the left term M and B is the type for N . For the context to be well formed, each of the $A \sqsubseteq B$ must be provable.

First, we add an axiom that \mathcal{U} is the least term of any type, to model the graduality property. Next, we add an axiom that \sqsubseteq is transitive, where both the value and effect type are allowed to vary simultaneously. The relation is reflexive as well, but this is admissible from congruence rules. Secondly, we give the congruence rules for functions and application, and the full system includes such a congruence rule for all term constructors. Next we have computation (β) and reasoning (η) rules for each type. For functions and if, these are standard call-by-value $\beta\eta$ rules, so we instead show only the handle rules. There are two β rules for handle. If the term being handled is a value, then the return clause is used. If the term being handled is a raise of an effect ε , it is equivalent to the handler clause $\phi(\varepsilon)$ where the continuation is the captured continuation surrounding the original handler term. We require this to be a let, but note that we have additional rules that imply that any evaluation context that doesn’t handle can be re-written as a let. We then have two reasoning (η) rules for handle. First, if M is handled by a handler with no effect clauses, then the handler is equivalent to a let-binding. Second, we have a rule that says that any clause that simply re-raises its operation with the same continuation it was passed can be dropped from the handler, as this is the same behavior as not catching the term at all. We next show rules describing the interaction of subtyping with value type casts, the full system includes analogous rules for effect types. The first says that an upcast followed by a subtyping coercion is less than a subtyping coercion followed by an upcast, and the downcast rule is similar. Finally, we have rules specifying the behavior of value and effect casts. These rules characterize upcasts as least upper bounds and downcasts as greatest lower bounds. The first rule shows that the downcast is a lower bound and the second that it is the greatest. The upcasts have similar rules, and we include analogous rules for effect casts as well.

4.2 Operational Semantics

Next, we show a selection of the rules of the operational semantics $M \mapsto M'$ in Figure 11, eliding the standard call-by-value rules for booleans, functions and let-bindings. We capture the left-to-right, call-by-value evaluation order by using evaluation contexts defined in Section 3.1. First, we have the β rules for handlers: when handling a value, execute the return clause. Next, when a raise occurs, we search for the closest enclosing handler that handles the raised effect and capture the intermediate evaluation context in the continuation passed to the appropriate handler. We capture this with the relation $E' \# \varepsilon$ which says that the evaluation context does not handle the given operation.

The next rules concern the behavior of effect casts. First, all effect casts are the identity on values. Next, when upcasting a raise, we re-raise the effect, but *upcast* the request and *downcast*

$$\begin{array}{c}
\frac{\Gamma \vdash_{\sigma} M : A}{\Gamma \vdash_{\sigma \sqsubseteq \sigma} \bar{U} \sqsubseteq M : A \sqsubseteq A} \\
\frac{\Gamma^{\sqsubseteq} \vdash_{\sigma_1 \sqsubseteq \sigma_2} M_1 \sqsubseteq M_2 : A_1 \sqsubseteq A_2 \quad \Gamma'^{\sqsubseteq} \vdash_{\sigma_2 \sqsubseteq \sigma_3} M_2 \sqsubseteq M_3 : A_2 \sqsubseteq A_3}{\Gamma''^{\sqsubseteq} \vdash_{\sigma_1 \sqsubseteq \sigma_3} M_1 \sqsubseteq M_3 : A_1 \sqsubseteq A_3} \\
\frac{\Gamma^{\sqsubseteq}, x : A \sqsubseteq A' \vdash_{\tau \sqsubseteq \tau'} M \sqsubseteq M' : B \sqsubseteq B'}{\Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \sigma'} \lambda x. M \sqsubseteq \lambda x. M' : A \rightarrow_{\tau} B \sqsubseteq A' \rightarrow_{\tau'} B'} \\
\frac{\Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \sigma'} M \sqsubseteq M' : A \rightarrow_{\sigma} B \sqsubseteq A' \rightarrow_{\sigma'} B' \quad \Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \sigma'} N \sqsubseteq N' : A \sqsubseteq A'}{\Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \sigma'} MN \sqsubseteq M'N' : B \sqsubseteq B'} \\
\text{handle } V \{ \text{ret } y.M \mid \phi \} \equiv M[y/v] \quad \text{handle } (\text{let } o = \text{raise } \varepsilon(x) \text{ in } N_k) \{ \text{ret } y.M \mid \phi \} \\
\equiv \phi(\varepsilon)[\lambda o. \text{handle } N_k \{ \text{ret } y.M \mid \phi \} / k] \\
\text{handle } M \{ \text{ret } x.N \mid \emptyset \} \equiv \text{let } x = M \text{ in } N \quad \frac{\forall \varepsilon \in \text{dom}(\phi). \psi(\varepsilon) = \phi(\varepsilon) \quad \forall \varepsilon \in \text{dom}(\psi). \varepsilon \notin \text{dom}(\phi) \Rightarrow \psi(\varepsilon) = k(\text{raise } \varepsilon(x))}{\text{handle } M \{ \text{ret } y.N \mid \phi \} \equiv \text{handle } M \{ \text{ret } y.N \mid \psi \} : \sigma!B} \\
\frac{A \leq: A' \quad B \leq: B' \quad \Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \sigma'} M \sqsubseteq N : A}{\Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \sigma'} \langle B \swarrow A \rangle M \sqsubseteq \langle B' \swarrow A' \rangle N : B'} \\
\frac{A \leq: A' \quad B \leq: B' \quad \Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \sigma'} M \sqsubseteq N : B}{\Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \sigma'} \langle A' \swarrow B' \rangle M \sqsubseteq \langle A \swarrow B \rangle N : A'} \\
\frac{\Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \sigma'} M \vdash_{\sigma} N : B}{\Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \sigma'} \langle A \swarrow B \rangle M \sqsubseteq N : A \sqsubseteq B} \\
\frac{\Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \sigma'} M \sqsubseteq N : A \sqsubseteq B}{\Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \sigma'} M \sqsubseteq \langle A \swarrow B \rangle N : A}
\end{array}$$

Fig. 10. Axiomatic Semantics

the response according to the types in the output effect type. An effect downcast works dually if the effect occurs in the result effect type. However, if the effect does not occur in the output effect type (which can only occur if the input effect type is ?), then an error is raised. Finally, we have the function downcast. Recall that a function cast applied to a value itself is a value, and only reduces when applied to a value. When this occurs in a downcast, as shown, the result reduces to applying the original function to an *upcasted* version of the input and *downcast* of the output, where this time we cast both value and effect types. Note the order of the value and effect casts on the output is arbitrarily chosen: because value casts only affect values and effect casts only affect effect operations, the two possible orders are equivalent. The elided cast for function upcasts is precisely dual, and finally there is a trivial cast rule for the identity cast on booleans.

We conclude the operational semantics with the following theorem, which establishes that the operational rules are in a sense *necessary* to any system that satisfies our axiomatic semantics (proof in the appendix).

THEOREM 4.1. *If $\vdash_{\emptyset} M, N : A$ and $M \mapsto N$ then $M \equiv N$ is provable in the axiomatic semantics.*

4.3 Subtyping, Gradual Subtyping and Coercions

The elaboration defined in Section 3.2 inserts casts of the form $\langle A \swarrow [A] \rangle \langle [B] \swarrow B \rangle M$ when a gradual subtyping $A \lesssim B$ is used in the type-checker. If we think of $[A]$ as the type of programs in the untracked language, this says to cast a program from one type to another, we should cast it to an untracked type and then to the other effect-tracking type, similar to prior work on cast calculi based on upcasts and downcasts [14]. This is a reasonable cast if we think of the untracked language as

$$\begin{array}{c}
E[\text{handle } V \{ \text{ret } x.N \mid \phi \}] \mapsto E[N[V/x]] \quad \frac{\epsilon \in \text{dom}(\phi) \quad E' \# \epsilon}{E[\text{handle } E'[\text{raise } (\epsilon @ A \rightsquigarrow B)(V)] \{ \text{ret } x.N \mid \phi \}] \mapsto E[\phi(\epsilon)[V/x][(\lambda y. \text{handle } (E'[y]) \{ \text{ret } x.N \mid \phi \})/k]]} \\
\\
E[(\lambda x.M)V] \mapsto E[M[V/x]] \quad E[\text{let } x = V \text{ in } M] \mapsto E[M[V/x]] \\
\\
E[\langle \sigma' \prec \sigma \rangle V] \mapsto E[V] \quad \frac{\epsilon @ A' \rightsquigarrow B' \in \sigma' \quad E' \# \epsilon}{E[\langle \sigma' \prec \sigma \rangle E'[\text{raise } (\epsilon @ A \rightsquigarrow B)(V)]] \mapsto E[\text{let } x = \langle B \leftarrow B' \rangle \text{raise } \epsilon(\langle A' \prec A \rangle V) \text{ in } \langle \sigma' \prec \sigma \rangle E'[x]]} \\
\\
E[\langle \sigma \leftarrow \sigma' \rangle V] \mapsto E[V] \quad \frac{\epsilon @ A \rightsquigarrow B \in \sigma \quad E' \# \epsilon}{E[\langle \sigma \leftarrow \sigma' \rangle E'[\text{raise } (\epsilon @ A' \rightsquigarrow B')(V)]] \mapsto E[\text{let } x = \langle B' \prec B \rangle \text{raise } \epsilon(\langle A \leftarrow A' \rangle V) \text{ in } \langle \sigma \leftarrow \sigma' \rangle E'[x]]} \\
\\
\frac{\epsilon @ A \rightsquigarrow B \notin \sigma \quad E' \# \epsilon}{E[\langle \sigma \leftarrow ? \rangle E'[\text{raise } (\epsilon @ A' \rightsquigarrow B')(V)]] \mapsto \mathcal{U}} \quad E[\langle (A \rightarrow_{\sigma} B) \leftarrow (A' \rightarrow_{\sigma'} B') \rangle V_f] \mapsto E[\langle B \leftarrow B' \rangle \langle \sigma \leftarrow \sigma' \rangle (V_f \langle A' \prec A \rangle V)]
\end{array}$$

Fig. 11. Operational semantics of Core GrEff

our “operational ground truth”, and so we should prove that any other translation is extensionally equivalent to this one. However, operationally, this can be quite a wasteful translation, as a cast can result in proxying at runtime, while *subtyping* coercions have no runtime behavior, and so are zero cost. For instance, if $A \lesssim B$ is true because in fact $A \leq B$, then there need not be any runtime cast at all. For this reason, we would prefer to optimize the cast based on the subtyping information in the proof of $A \lesssim B$. Since A may be more imprecise than B in some subterms and vice-versa, the structure of the cast should still be an upcast followed by a downcast, but with the possibility that we use implicit subtyping coercions at some points. There are three places we might insert the implicit subtyping coercion: before the upcast, between the upcast and downcast and after the downcast. From the proof of $A \lesssim A'$, we can extract types and subtyping/precision derivations as in Figure 12.

On the left we have a “pure subtyping” component of the gradual subtyping proof coming from A , and on the right we we have the pure subtyping component coming from B . In the middle we have two “dynamic” types also related by subtyping. There are then three paths from A to A' in this diagram, which generate three different potential casts with implicit subtyping coercions ensuring they are well-typed as taking A to A' : (1) Up and then right twice $\langle B \leftarrow D_h \rangle \langle D_h \prec A_h \rangle$ (2) Right, up and then right: $\langle B \leftarrow D_h \rangle \langle D_l \prec A \rangle$ (3) Right twice and then up: $\langle B_l \leftarrow D_l \rangle \langle D_l \prec A \rangle$ Fortunately we can choose whichever is

$$\begin{array}{ccccc}
A_h & \sqsubseteq & D_h & \supseteq & B \\
\ddot{v}_i & & \ddot{v}_i & & \ddot{v}_i \\
A & \sqsubseteq & D_l & \supseteq & B_l
\end{array}$$

Fig. 12. Situation derivable from $A \lesssim B$

operationally preferable: each of these casts is equivalent as a function from A to B and they are all equivalent to the ground truth cast $\langle A' \leftarrow [A] \rangle \langle [A] \prec A \rangle$. The above discussion applies equally well to effect casts, which are even simpler in that the “ground-truth” always factors through the a single most imprecise effect type: the dynamic effect type.

5 SOUNDNESS AND GRADUALITY

In this section we establish that the axiomatic semantics of core GrEff has a sound model in terms of its operational semantics. This establishes two key properties: equivalent terms ($M \equiv N$) are contextually equivalent in the operational semantics, and the graduality property holds. First, we review the definition of the graduality property, and then we give a logical relations model and prove that any provable inequality $M \sqsubseteq N$ implies that the terms are related in the logical relation.

5.1 Static and Dynamic Gradual Guarantees

GrEff is designed to support a smooth *migration* from imprecise to precise typing. The static gradual guarantee [21] formalizes a syntactic element of this idea of a smooth migration. The static gradual guarantee informally says that increasing the precision of type annotations on a program can only make it *harder* to satisfy the static type checker, or viewed the other way around, decreasing the precision of type annotations can only make it *easier* to satisfy the static type checker. Then the dynamic gradual guarantee, also known as *graduality*, establishes the semantic counterpart: increasing the precision of type annotations on a program should only make it *harder* to terminate without a *dynamic* type error, and furthermore except where there are dynamic type errors, the behavior of the program should match the original. These properties can be formalized as a form of *monotonicity* of the elaboration of the syntactic programs of surface GrEff into the semantically meaningful core GrEff programs as follows. First, we define a syntactic term precision ordering \sqsubseteq^{syn} on untyped GrEff programs as the congruence closure of the type precision ordering. Then the static gradual guarantee says that this is a monotone *partial* function:

THEOREM 5.1 (STATIC GRADUAL GUARANTEE). *If $P \sqsubseteq^{\text{syn}} P'$, then if $\cdot \vdash P \Rightarrow \Sigma \vdash_{\sigma} M : A$, then there exist M', σ', A' such that $\cdot \vdash P' \Rightarrow \Sigma \vdash_{\sigma'} M' : A'$ such that $\Sigma \vdash M \sqsubseteq M' : (\sigma \sqsubseteq \sigma') ! (A \sqsubseteq A')$*

Then the dynamic gradual guarantee says that this extends to monotonicity in the following *semantic* ordering on core GrEff terms:

Definition 5.2 (Error Ordering on Closed Programs). Given $\cdot \vdash_{\emptyset} M, M' : \text{bool}$, define $M \sqsubseteq^{\text{sem}} M'$ to hold when one of the following is satisfied (1) $M \mapsto^* \perp$, (2) $M \uparrow$ and $M' \uparrow$, (3) $M \mapsto^* \text{true}$ and $M' \mapsto^* \text{true}$ (4) $M \mapsto^* \text{false}$ and $M' \mapsto^* \text{false}$.

THEOREM 5.3 (DYNAMIC GRADUAL GUARANTEE). *If $\Sigma \mid \cdot \vdash_{\emptyset \sqsubseteq \emptyset} M \sqsubseteq M' : \text{bool}$, then $M \sqsubseteq^{\text{sem}} M'$.*

This theorem is stated in terms of closed terms of a fixed type, but to prove it we need a stronger inductive hypothesis, i.e., the logical relation for open terms. The resulting theorem that any inequation provable in the theory implies the semantic ordering is called *graduality*, as it is analogous in structure to the *parametricity* theorem in parametric polymorphism. Then the dynamic gradual guarantee follows as a corollary.

5.2 Logical Relation

In Figure 13, we present the definition of the step-indexed logical relation for graduality. Following prior work on logical relations for graduality, the relation is indexed not by types, but by *derivations* of type precision facts, i.e., proof terms for $c : A \sqsubseteq B$ or $d_{\sigma} : \sigma \sqsubseteq \sigma'$. We present the definition of these proof terms in the appendix. For a type precision derivation d , define d^l and d^r to be the types such that $d : d^l \sqsubseteq d^r$, and analogously for effect types.

Many of the details are similar to prior work, especially [15], so we highlight the aspects that are novel: the handling of effect types and subtyping. First, our logical relation refers to effect types in addition to value types. Next, in addition to the usual expression and value relations, we have also a *result* relation and a *continuation* relation. In our language, a **result** is either a

$$\begin{array}{lcl}
(M_1, M_2) \in (\blacktriangleright R)_j & \iff & j = 0 \vee (j = k + 1 \wedge (M_1, M_2) \in R_k) \\
(V_1, V_2) \in \mathcal{V}_j^\sim \llbracket \text{bool} \rrbracket & \iff & (V_1, V_2) \in \text{VAtom bool} \wedge (V_1 = V_2 = \text{true}) \vee (V_1 = V_2 = \text{false}) \\
(V_1, V_2) \in \mathcal{V}_j^\sim \llbracket d_i \rightarrow_{d_\sigma} d_o \rrbracket & \iff & (V_1, V_2) \in \text{VAtom } d_i \rightarrow_{d_\sigma} d_o \wedge \\
& & \forall k \leq j. \forall (V_{i1}, V_{i2}) \in \mathcal{V}_k^\sim \llbracket d_i \rrbracket. \\
& & (V_1 V_{i1}, V_2 V_{i2}) \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket (\mathcal{V}^\sim \llbracket d_o \rrbracket) \\
(M_1, M_2) \in \mathcal{E}_j^\leq \llbracket d_\sigma \rrbracket (R, A^l, A^r) & \iff & (M_1, M_2) \in \text{TAtom } A^l A^r d_\sigma \wedge (M_1 \mapsto^{j+1} \\
& & \vee (\exists k \leq j. (M_1 \mapsto^{j-k} \mathcal{U})) \\
& & \vee (\exists (N_1, N_2) \in \mathcal{R}_k^\leq \llbracket d_\sigma \rrbracket R \wedge M_1 \mapsto^{j-k} N_1 \wedge M_2 \mapsto^* N_2))) \\
(M_1, M_2) \in \mathcal{E}_j^\geq \llbracket d_\sigma \rrbracket (R, A^l, A^r) & \iff & (M_1, M_2) \in \text{TAtom } A^l A^r d_\sigma \wedge (M_2 \mapsto^{j+1} \\
& & \vee (\exists k \leq j. (M_2 \mapsto^{j-k} \mathcal{U} \wedge M_1 \mapsto^* \mathcal{U})) \\
& & \vee (\exists N_2. M_2 \mapsto^{j-k} N_2 \wedge M_1 \mapsto^* \mathcal{U})) \\
& & \vee (\exists (N_1, N_2) \in \mathcal{R}_k^\geq \llbracket d_\sigma \rrbracket R \wedge M_2 \mapsto^{j-k} N_2 \wedge M_1 \mapsto^* N_1))) \\
(M_1, M_2) \in \mathcal{R}_j^\sim \llbracket d_\sigma \rrbracket (R, A^l, A^r) & \iff & (M_1, M_2) \in \text{TAtom } A^l A^r d_\sigma \wedge \\
& & ((\text{val}(M_1) \wedge \text{val}(M_2) \wedge (M_1, M_2) \in R_j) \\
& & \vee (\exists \epsilon : c \rightsquigarrow d \in d_\sigma, E^l \# \epsilon, E^r \# \epsilon, V^l, V^r. \\
& & (V^l, V^r) \in (\blacktriangleright \mathcal{V}^\sim \llbracket c \rrbracket)_j \wedge \\
& & (x^l.E^l[x^l], x^r.E^r[x^r]) \in (\blacktriangleright \mathcal{K}^\sim \llbracket d \rrbracket)_j \\
& & (\mathcal{E}^\sim \llbracket d_\sigma \rrbracket (R, A^l, A^r), (d_\sigma^l ! A^l), (d_\sigma^r ! A^r)) \wedge \\
& & M_1 = E^l[\text{raise } \epsilon(V^l)] \wedge M_2 = E^r[\text{raise } \epsilon(V^r)])) \\
(x^l.M^l, x^r.M^r) \in & \iff & (M^l, M^r) \in \text{EctxAtom } c (\sigma^l ! A^l) (\sigma^r ! A^r) \wedge \\
\mathcal{K}_j^\sim \llbracket c \rrbracket (S, (\sigma^l ! A^l), (\sigma^r ! A^r)) & \iff & \forall k \leq j. (V^l, V^r) \in \mathcal{V}_k^\sim \llbracket c \rrbracket. (M^l[V^l/x^l], M^r[V^r/x^r]) \in S_k \\
(\gamma_1, \gamma_2) \in \mathcal{G}_j^\sim \llbracket \Gamma^\square \rrbracket & \iff & \forall (x_1 \sqsubseteq x_2 : c) \in \Gamma^\square. (\gamma_1(x_1), \gamma_2(x_2)) \in \mathcal{V}_j^\sim \llbracket c \rrbracket
\end{array}$$

Fig. 13. Logical Relation

value, or an evaluation context E wrapping a raise of an effect ϵ , such that E is apart from ϵ . The result relation specifies the conditions for two such results to be related. Notable here is the use of the later modality when specifying when two wrapped raise terms are related. The intuition is that we only require the values being raised, and the evaluation contexts wrapping the raises, to be related one step later. Finally, the relations are parameterized by precision derivations. In the case of the expression and result relations, this is an effect precision derivation, while for values and continuations, it is a value type precision derivation. This is analogous to the usual approach whereby logical relation is indexed by a type. But instead of using types, we use precision derivations, i.e., the proof that the LHS term is more precise than the type of the RHS term.

As in previous work on logical relations for graduality, the expression logical relation $\mathcal{E}^\sim \llbracket \cdot \rrbracket$ is split into two relations $\mathcal{E}^\leq \llbracket \cdot \rrbracket$ and $\mathcal{E}^\geq \llbracket \cdot \rrbracket$. The former counts the steps taken by the left-hand term, while the latter counts steps taken by the right-hand term. Despite needing two relations, we are for the most part able to abstract over their differences: most of the lemmas we prove hold for both relations with no adjustment needed. Notable exceptions are transitivity and the anti- and forward reduction lemmas: these lemmas make crucial use of step counting, so naturally the side whose steps we are counting makes a difference.

Given a step-indexed relation R , we define an operator $\blacktriangleright R$ (pronounced “later R ”) as follows: Terms M_1 and M_2 are related in $\blacktriangleright R$ at index n if and only if either n is zero, or $n \geq 1$ and M_1 and M_2 are related in R at index $n - 1$.

One novel aspect of our logical relation is the result relation $\mathcal{R} \sim \llbracket \cdot \rrbracket$. This relation relates terms M_1 and M_2 – of type A^l and A^r respectively – representing either two values or two “evaluations” of raised operations. The relation is parameterized by a step-indexed relation R between *values* of type A^l and A^r (the types of M_1 and M_2). M_1 and M_2 are related by $\mathcal{R} \sim \llbracket \cdot \rrbracket$ when either (1) both terms are values and are related by R at the appropriate step index, or (2) there exists an effect ϵ in d_σ , values related later, and evaluation contexts (i.e., continuations – see below) related later, such that M_1 is equal to raising the effect and then wrapping it in the continuation, and likewise for M_2 .

The relation $\mathcal{K} \sim \llbracket \cdot \rrbracket$ relates evaluation contexts E_1 and E_2 representing continuations that accept values. The evaluation contexts each have a hole \bullet – the type of the hole is the type of the input value to the continuation. To enforce that the continuations accept *values* only, and not arbitrary terms, the inputs to the continuation relation are actually *terms* M^l and M^r with free variables x^l and x^r , respectively. E_1 and E_2 also have “output” types (A^l and A^r) and “output” effect sets (σ^l and σ^r). When values are plugged into E_1 and E_2 , the result is two terms having types A^l and A^r and effect sets σ^l and σ^r , respectively.

5.3 Proof of Graduality

Our goal is to prove that the inequational theory is sound with respect to the logical relation. First we define the notion of two terms being related semantically:

$$\Gamma \sqsubseteq \vdash_{d_\sigma} M_1 \sqsubseteq M_2 \in c := \forall \sim \in \{<, >\}. \forall j \in \mathbb{N}. \forall (\gamma_1, \gamma_2) \in \mathcal{V}_j \llbracket \Gamma \rrbracket. (M_1[\gamma_1], M_2[\gamma_2]) \in \mathcal{E}_j \llbracket d_\sigma \rrbracket \mathcal{V} \sim \llbracket c \rrbracket.$$

That is, M_1 and M_2 are related if for all j and all substitutions of values γ_1 and γ_2 related at j , the resulting terms are related in $\mathcal{E}_j \llbracket d_\sigma \rrbracket \mathcal{V} \sim \llbracket c \rrbracket$, where this needs to hold both when \sim is $<$ and when it is $>$. Our goal is then to prove the following:

THEOREM 5.4 (GRADUALITY). *If $\Gamma \sqsubseteq \vdash_{d_\sigma} M \sqsubseteq N : c$ then $\Gamma \sqsubseteq \vdash_{d_\sigma} M \sqsubseteq N \in c$*

We provide here a high-level overview of the proof; the complete proofs are in the appendix. We begin by establishing lemmas that are used in the remaining proofs. Noteworthy here are the reduction lemmas (anti- and forward-reduction), and the monadic bind lemma. These lemmas are largely standard, having appeared in previous work on step-indexed logical relations. However, our monadic bind lemma must also deal with the case where the terms in question are related raises. Another important principle is that of Löb induction (Lemma D.16), which states that if from the assumption that a proposition holds “later” it follows that it holds now, then the proposition holds now. This boils down to induction on the step index, but it is convenient to work with the later modality as it eliminates much of the tedious reasoning about step indexing. Löb induction is used in proofs involving handling a raised effect, as well as those involving an effect cast, as these are the cases of the operational semantics for which step indexing is necessary in order to be well-founded.

With these lemmas, we first prove soundness of each of the congruence rules for term precision. All of the proofs are straightforward uses of the monadic bind lemma along with the reduction lemmas. Next, we prove soundness of the rules of the equational theory, e.g., the β and η laws, and transitivity. Finally, we prove soundness of the rules for casts and subtyping.

6 DISCUSSION

Prior Work on Gradual Effects. The most significant prior work on gradual effects is Bañados Schwerter et al. [1], which defined a gradual effect system based on the generic effect calculus of Marino and Millstein [13] using an early version of the *abstracting gradual typing* (AGT) framework for gradual type systems [7]. While we based GrEff on effect handlers rather than the generic effect calculus, there are significant similarities in the typing: function types and typing judgments are indexed by a set of effect operations in each system. The most significant difference is that

their framework is parameterized by a fixed effect theory, whereas GrEff has explicit support for declaration of new effects in the program. They additionally support a “gradual row” effect which tracks that some effects a term raised are known only to ever use certain types but there is no information on other effects. The presence of this type does not change the semantics of runtime casts, but may be useful to express certain optimizations.

Another related area of research is on gradual typing with delimited continuations, which are mutually expressible with effect handlers [5, 18]. Takikawa et al ([25]) propose a gradual type system and semantics via contracts for a language with delimited continuations using typed prompts. They consider only value types and untracked function types that do not say which prompts are expected to be present. They show that a naive contract based implementation is unsound because a dynamically typed program can interact with a typed prompt and therefore the prompts themselves must be equipped with contracts, even though it does not correspond to any value being imported. In core GrEff, this unsoundness is ruled out by using intrinsic typing: the problem corresponds to raising an effect operation with a different type than the type expected by the closest handler, which is precisely what the effect type system tracks. Wrapping the prompt in contracts is behaviorally equivalent to what is achieved by our effect type casts. Sekiyama, Ueda and Igarashi present a blame calculus for a language with shift and reset [20]. The blame calculus is analogous to our core GrEff language, and uses a type and effect system for the answer types of shift/reset. They do not develop a surface language that elaborates to this blame calculus like our GrEff, and there is no analogue of effect operations in shift/reset-based systems so there are no nominal aspects of their language. Additionally, while they have an effect system to keep track of answer types, they do not have effect casts.

Prior Approaches to Gradual Nominal Datatypes. We are also not the first to consider the combination of gradual and nominal typing. The closest match to our design is in Typed Racket’s support for typed structs. In Racket, a struct is a kind of record type that (by default) is generative in that it creates a new type tag distinct from all others. Typed Racket supports import of untyped Racket structs into Typed Racket, where types are assigned to the fields, and values of the struct type are then wrapped in contracts accordingly. This is quite close to our treatment of nominal effect operations which can be thought of as adding new cases to the dynamic effect monad rather than dynamic type. Our type system is more complex however, since in our system modules can use dynamically typed effects whereas in Typed Racket, there is no syntactic type for dynamically typed values, when imported into typed code the system must give a completely precise type. Malewski et al ([12]) present a design for gradual typing with nominal algebraic datatypes. Their focus is on the gradual migration from datatypes whose cases are open-ended to datatypes with a fixed set of constructors. They do not consider use-case we have where different modules have different typings for the same nominal constructor.

Extensions. GrEff is a simplified model of typed algebraic effects, and there are several natural extensions that would be useful to model. A natural extension would be to include *recursive* effect types, which would allow us to provide precise static types for our example of a threading library where any forked threads can freely fork additional threads itself, which in GrEff is only possible to arbitrary depth using dynamic effect typing. Additionally, many typed functional languages that we may want to extend with effect typing in this way use Hindley-Milner-style polymorphic type schemes, which GrEff does not support. Possibly we can build on previous work for gradual typing in unification-based type systems[6, 23].

Another simplification in the design of GrEff is that it supports only *static* declarations of effect operations at the top-level of a module. In contrast, SML supports dynamic creation of new exceptions at runtime, so that the exception type is a kind of runtime-extensible sum type.

Interesting future work would be to modify prior graduality proofs for runtime-extensible dynamic to support runtime effect declaration [15].

REFERENCES

- [1] Felipe Bañados Schwerter, Ronald Garcia, and Éric Tanter. 2014. A Theory of Gradual Effect Systems. In *Proceedings of the 19th ACM SIGPLAN International Conference on Functional Programming* (Gothenburg, Sweden) (ICFP '14). 283–295.
- [2] Jonathan Immanuel Brachthäuser, Philipp Schuster, and Klaus Ostermann. 2020. Effekt: Capability-passing style for type- and effect-safe, extensible effect handlers in Scala. *J. Funct. Program.* 30 (2020), e8. <https://doi.org/10.1017/S0956796820000027>
- [3] WasmFX Contributors. [n.d.]. WasmFX: Effect Handlers for WebAssembly. <https://wasmfx.dev/> Accessed: 2020-11-10.
- [4] Ezra Cooper, Sam Lindley, Philip Wadler, and Jeremy Yallop. 2006. Links: Web Programming Without Tiers. In *Formal Methods for Components and Objects, 5th International Symposium, FMCO 2006, Amsterdam, The Netherlands, November 7-10, 2006, Revised Lectures (Lecture Notes in Computer Science, Vol. 4709)*. 266–296. https://doi.org/10.1007/978-3-540-74792-5_12
- [5] Yannick Forster, Ohad Kammar, Sam Lindley, and Matija Pretnar. 2019. On the expressive power of user-defined effects: Effect handlers, monadic reflection, delimited control. *J. Funct. Program.* 29 (2019), e15. <https://doi.org/10.1017/S0956796819000121>
- [6] Ronald Garcia and Matteo Cimini. 2015. Principal Type Schemes for Gradual Programs. In *Proceedings of the 42nd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2015, Mumbai, India, January 15-17, 2015*, Sriram K. Rajamani and David Walker (Eds.). ACM, 303–315. <https://doi.org/10.1145/2676726.2676992>
- [7] Ronald Garcia, Alison M. Clark, and Éric Tanter. 2016. Abstracting Gradual Typing. In *ACM Symposium on Principles of Programming Languages (POPL)*.
- [8] Oleg Kiselyov, Amr Sabry, and Cameron Swords. 2013. Extensible effects: an alternative to monad transformers. In *Proceedings of the 2013 ACM SIGPLAN Symposium on Haskell, Boston, MA, USA, September 23-24, 2013*. ACM, 59–70. <https://doi.org/10.1145/2503778.2503791>
- [9] Nico Lehmann and Éric Tanter. 2017. Gradual Refinement Types. In *ACM Symposium on Principles of Programming Languages (POPL)*.
- [10] Daan Leijen. 2014. Koka: Programming with Row Polymorphic Effect Types. In *Proceedings 5th Workshop on Mathematically Structured Functional Programming, MSFP@ETAPS 2014, Grenoble, France, 12 April 2014 (EPTCS, Vol. 153)*. 100–126. <https://doi.org/10.4204/EPTCS.153.8>
- [11] Sam Lindley, Conor McBride, and Craig McLaughlin. 2017. Do be do be do. In *Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming Languages, POPL 2017, Paris, France, January 18-20, 2017*. ACM, 500–514. <https://doi.org/10.1145/3009837.3009897>
- [12] Stefan Malewski, Michael Greenberg, and Éric Tanter. 2021. Gradually structured data. *Proc. ACM Program. Lang.* 5, OOPSLA (2021), 1–29. <https://doi.org/10.1145/3485503>
- [13] Daniel Marino and Todd D. Millstein. 2009. A generic type-and-effect system. In *Proceedings of TLDI'09: 2009 ACM SIGPLAN International Workshop on Types in Languages Design and Implementation, Savannah, GA, USA, January 24, 2009*, Andrew Kennedy and Amal Ahmed (Eds.). ACM, 39–50. <https://doi.org/10.1145/1481861.1481868>
- [14] Max S. New and Amal Ahmed. 2018. Graduality from Embedding-Projection Pairs. In *International Conference on Functional Programming (ICFP), St. Louis, Missouri*.
- [15] Max S. New, Dustin Jamner, and Amal Ahmed. 2020. Graduality and parametricity: together again for the first time. *Proc. ACM Program. Lang.* 4, POPL (2020), 46:1–46:32. <https://doi.org/10.1145/3371114>
- [16] Max S. New and Daniel R. Licata. 2018. Call-by-name Gradual Type Theory. In *Formal Structures for Computation and Deduction, Oxford England*.
- [17] Max S. New, Daniel R. Licata, and Amal Ahmed. 2019. Gradual Type Theory. In *ACM Symposium on Principles of Programming Languages (POPL), Cascais, Portugal*.
- [18] Maciej Piróg, Piotr Polesiuk, and Filip Sieczkowski. 2019. Typed Equivalence of Effect Handlers and Delimited Control. In *4th International Conference on Formal Structures for Computation and Deduction, FSCD 2019, June 24-30, 2019, Dortmund, Germany (LIPIcs, Vol. 131)*, Herman Geuvers (Ed.). Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 30:1–30:16. <https://doi.org/10.4230/LIPIcs.FSCD.2019.30>
- [19] Gordon D. Plotkin and Matija Pretnar. 2009. Handlers of Algebraic Effects. In *Programming Languages and Systems, 18th European Symposium on Programming, ESOP 2009, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2009, York, UK, March 22-29, 2009. Proceedings (Lecture Notes in Computer Science, Vol. 5502)*. 80–94. https://doi.org/10.1007/978-3-642-00590-9_7
- [20] Taro Sekiyama, Soichiro Ueda, and Atsushi Igarashi. 2015. Shifting the Blame - A Blame Calculus with Delimited Control. In *Programming Languages and Systems - 13th Asian Symposium, APLAS 2015, Pohang, South Korea, November*

- 30 - December 2, 2015, *Proceedings (Lecture Notes in Computer Science, Vol. 9458)*, Xinyu Feng and Sungwoo Park (Eds.). Springer, 189–207. https://doi.org/10.1007/978-3-319-26529-2_11
- [21] Jeremy Siek, Micahel Vitousek, Matteo Cimini, and John Tang Boyland. 2015. Refined Criteria for Gradual Typing. In *1st Summit on Advances in Programming Languages (SNAPL 2015)*.
- [22] Jeremy G. Siek and Walid Taha. 2006. Gradual Typing for Functional Languages. In *Scheme and Functional Programming Workshop (Scheme)*. 81–92.
- [23] Jeremy G. Siek and Manish Vachharajani. 2008. Gradual typing with unification-based inference. In *Proceedings of the 2008 Symposium on Dynamic Languages, DLS 2008, July 8, 2008, Paphos, Cyprus*, Johan Brichau (Ed.). ACM, 7. <https://doi.org/10.1145/1408681.1408688>
- [24] K. C. Sivaramakrishnan, Stephen Dolan, Leo White, Tom Kelly, Sadiq Jaffer, and Anil Madhavapeddy. 2021. Retrofitting effect handlers onto OCaml. In *PLDI '21: 42nd ACM SIGPLAN International Conference on Programming Language Design and Implementation, Virtual Event, Canada, June 20-25, 2021*. ACM, 206–221. <https://doi.org/10.1145/3453483.3454039>
- [25] Asumu Takikawa, T. Stephen Strickland, and Sam Tobin-Hochstadt. 2013. Constraining Delimited Control with Contracts. In *Programming Languages and Systems - 22nd European Symposium on Programming, ESOP 2013, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2013, Rome, Italy, March 16-24, 2013. Proceedings (Lecture Notes in Computer Science, Vol. 7792)*, Matthias Felleisen and Philippa Gardner (Eds.). Springer, 229–248. https://doi.org/10.1007/978-3-642-37036-6_14
- [26] Sam Tobin-Hochstadt and Matthias Felleisen. 2008. The Design and Implementation of Typed Scheme. In *ACM Symposium on Principles of Programming Languages (POPL), San Francisco, California*.

$$\begin{array}{c}
\Sigma \vdash \text{bool} : \text{bool} \sqsubseteq \text{bool} \qquad \frac{\Sigma \vdash d_i : A \sqsubseteq A' \quad \Sigma \vdash d_e : \sigma \sqsubseteq \sigma' \quad \Sigma \vdash d_o : B \sqsubseteq B'}{\Sigma \vdash d_i \rightarrow_{d_e} d_o : A \rightarrow_{\sigma} B \sqsubseteq A' \rightarrow_{\sigma'} B'} \\
\\
\Sigma \vdash ? : ? \sqsubseteq ? \qquad \frac{\begin{array}{c} \text{supp}(d_c) = \text{supp}(\sigma_c) = \text{supp}(\sigma'_c) \\ (\forall \varepsilon : c \rightsquigarrow d \in d_c, \varepsilon : A \rightsquigarrow B \in \sigma_c, \varepsilon : A' \rightsquigarrow B' \in \sigma'_c. \\ \Sigma \vdash c : A \sqsubseteq A' \quad \Sigma \vdash B \sqsubseteq B') \end{array}}{\Sigma \vdash d_c : \sigma_c \sqsubseteq \sigma'_c} \qquad \frac{\Sigma \vdash d_c : \sigma_c \sqsubseteq \Sigma |_{\text{supp}(\sigma_c)}}{\Sigma \vdash \text{inj}(d_c) : \sigma_c \sqsubseteq ?} \\
\\
\frac{\varepsilon : c \rightsquigarrow d \in \Sigma}{\Sigma \vdash \varepsilon : c \rightsquigarrow d \in ?} \qquad \frac{\Sigma \vdash \varepsilon : c' \rightsquigarrow d' \in d_c \quad c = \text{inj}(c') \quad d = \text{inj}(d')}{\Sigma \vdash \varepsilon : c \rightsquigarrow d \in \text{inj}(d_c)}
\end{array}$$

Fig. 14. Type and Effect Precision Derivations

A SYNTAX AND ELABORATION

We give a term assignment for effect precision in Figure 14. In it we use the notion of an effect operation being in a precision derivation $\varepsilon : c \rightsquigarrow d \in d_c$. For when d_c itself is a partial function this is just as with earlier usage, but when $d_c = ?$ or $d_c = \text{inj}(d'_c)$ we use the definition at the bottom of the figure.

Though the generating axioms are different from the simple presentation in the body of the paper, we show that provability is not affected:

LEMMA A.1 (CORRECTNESS OF TERM ASSIGNMENT). *Assuming $\Sigma \vdash A$ and $\Sigma \vdash B$, the following are equivalent*

- $A \sqsubseteq A'$ is provable in the system in Figure 5
- There exists a derivation $\Sigma \vdash c : A \sqsubseteq A'$ in the system in Figure 14.

Similarly, assuming $\Sigma \vdash \sigma$ and $\Sigma \vdash \sigma'$, the following are equivalent

- $\sigma \sqsubseteq \sigma'$ is provable in the system in Figure 5
- There exists a derivation $\Sigma \vdash c_e : \sigma \sqsubseteq \sigma'$ in the system in Figure 14.

Next we define gradual join and meet of value and effect types in Figure 15. Note that the definition is quite simple for concrete effect sets because this is only used on effects *within* the same module, so we never have to consider the case where the two sides assign different effects to the same operation name ε .

Now we define a notion of subtyping of precision derivations, which will be needed in the proofs involving the interaction between subtyping and casts.

B (IN)EQUATIONAL THEORY

In this section we describe the full inequational theory and then prove several derivable theorems in the theory.

Note that for brevity, we use some shorthands: rather than writing out the full $\Sigma \mid \Gamma^{\sqsubseteq} \vdash_{\sigma \sqsubseteq \tau} M \sqsubseteq N : A \sqsubseteq B$, (1) we elide $\Sigma \mid \Gamma^{\sqsubseteq}$, and all rules should be interpreted as holding under an arbitrary such contexts (2) rather than write $\sigma \sqsubseteq \tau$ and $A \sqsubseteq B$, we use instead precision derivations d_{σ} , c and (3) whenever it is clear, we elide the types as well, especially for equational rules.

$$\begin{aligned}
& \text{bool} \tilde{\vee} \text{bool} = \text{bool} \\
& (A \rightarrow_{\sigma} B) \tilde{\vee} (A' \rightarrow_{\sigma'} B') = (A \tilde{\wedge} A') \rightarrow_{\sigma \tilde{\vee} \sigma'} (B \tilde{\vee} B') \\
& \quad ? \tilde{\vee} \sigma = \sigma \\
& \quad \sigma \tilde{\vee} ? = \sigma \\
& \quad \sigma_c \tilde{\vee} \tau_c = \sigma_c \cup \tau_c \\
\\
& \text{bool} \tilde{\wedge} \text{bool} = \text{bool} \\
& (A \rightarrow_{\sigma} B) \tilde{\wedge} (A' \rightarrow_{\sigma'} B') = (A \tilde{\vee} A') \rightarrow_{\sigma \tilde{\wedge} \sigma'} (B \tilde{\wedge} B') \\
& \quad ? \tilde{\wedge} \sigma = ? \\
& \quad \sigma \tilde{\wedge} ? = ? \\
& \quad \sigma_c \tilde{\wedge} \tau_c = \sigma_c \cap \tau_c
\end{aligned}$$

Fig. 15. Gradual Join and Meet

$$\begin{array}{c}
\text{bool} \leq: \text{bool} \quad \frac{d_i \leq: c_i \quad c_e \leq: d_e \quad c_o \leq: d_o}{c_i \rightarrow_{c_e} c_o \leq: d_i \rightarrow_{d_e} d_o} \quad ? \leq: ? \quad \frac{c \leq: d}{\text{inj}(c) \leq: \text{inj}(d)} \\
\\
\frac{\forall \varepsilon : c \rightsquigarrow d \in d_c \cdot \varepsilon : c' \rightsquigarrow d' \in d'_c \wedge c \leq: c' \wedge d' \leq: d}{d_c \leq: d'_c} \quad \frac{c \leq: \text{inj}(\Sigma)}{c \leq: ?} \quad \frac{c \leq: d}{c \leq: \text{inj}(d)}
\end{array}$$

Fig. 16. Subtyping of Precision Derivations

First we need general call-by-value reasoning principles.

$$\frac{M[x : A] \equiv N[x : A] \quad V \equiv V' : A}{M[V/x] \equiv N[V'/x]} \text{VALSUBST} \quad \text{let } x = y \text{ in } N \equiv N[y/x] \text{ MONADUNITL}$$

$$\text{let } x = M \text{ in } x \equiv M \text{ MONARDUNITR}$$

$$\text{let } y = (\text{let } x = M \text{ in } N) \text{ in } P \equiv \text{let } x = M \text{ in let } y = N \text{ in } P \text{ MONADASSOC}$$

$$M[x : \text{bool}] \equiv \text{if } x \{M[\text{true}/x]\} \{M[\text{false}/x]\} \text{ BOOLETA}$$

$$\text{if true}\{N_t\}\{N_f\} \equiv N_t \text{ BOOLBETATRU} \quad \text{if false}\{N_t\}\{N_f\} \equiv N_f \text{ BOOLBETAFALSE}$$

$$\text{if } M\{N_t\}\{N_f\} \equiv \text{let } x = M \text{ in if } x\{N_t\}\{N_f\} \text{ IFEVAL} \quad (\lambda x.M)V \equiv M[V/x] \text{ FUNBETA}$$

$$(V : A \rightarrow B) \equiv \lambda x.Vx \text{ FUNETA} \quad MN \equiv \text{let } x = M \text{ in let } y = N \text{ in } xy \text{ APPEVAL}$$

Next, the rules specifically for raise and handlers:

$$\text{handle } x \{ \text{ret } y.M \mid \phi \} \equiv M[x/y] \text{ HANDLEBETARET}$$

$$\text{handle } (\text{let } o = \text{raise } \varepsilon(x) \text{ in } N_k) \{ \text{ret } y.M \mid \phi \} \equiv \frac{\phi(\varepsilon)[\lambda o.\text{handle } N_k \{ \text{ret } y.M \mid \phi \}/k]}{\text{HANDLEBETARAISE}}$$

$$\text{raise } \varepsilon(M) \equiv \text{let } x = M \text{ in raise } \varepsilon(x) \text{ RAISEEVAL}$$

$$\text{handle } M \{ \text{ret } x.N \mid \emptyset \} \equiv \text{let } x = M \text{ in } N \text{ HANDLEEMPTY}$$

$$\frac{\forall \varepsilon \in \text{dom}(\phi). \psi(\varepsilon) = \phi(\varepsilon) \quad \forall \varepsilon \in \text{dom}(\psi). \varepsilon \notin \text{dom}(\phi) \Rightarrow \psi(\varepsilon) = k(\text{raise } \varepsilon(x))}{\text{handle } M \{ \text{ret } y.N \mid \phi \} \equiv \text{handle } M \{ \text{ret } y.N \mid \psi \}} \text{HANDLEEXT}$$

Next, the congruence rules

$$\frac{x_1 \sqsubseteq x_2 : c \in \Gamma^\square}{\Sigma \mid \Gamma^\square \vdash_{d_\sigma} x_1 \sqsubseteq x_2 : c} \text{VARCONG} \quad \frac{}{\vdash_{d_\sigma} \text{true} \sqsubseteq \text{true} : \text{bool}} \text{TRUECONG}$$

$$\frac{}{\vdash_{d_\sigma} \text{false} \sqsubseteq \text{false} : \text{bool}} \text{FALSECONG} \quad \frac{x_1 \sqsubseteq x_2 : c \vdash_{d_{\sigma'}} M_1 \sqsubseteq M_2 : d}{\vdash_{d_\sigma} \lambda x_1.M_1 \sqsubseteq \lambda x_2.M_2 : c \rightarrow_{d_{\sigma'}} d} \text{LAMBDACONG}$$

$$\frac{\vdash_{d_\sigma} M_1 \sqsubseteq M_2 : c \rightarrow_{d_\sigma} d \quad \vdash_{d_\sigma} N_1 \sqsubseteq N_2 : c}{\vdash_{d_\sigma} M_1 N_1 \sqsubseteq M_2 N_2 : d} \text{APPCONG}$$

$$\frac{\vdash_{d_\sigma} M_1 \sqsubseteq M_2 : c \quad x_1 \sqsubseteq x_2 : c \vdash_{d_\sigma} N_1 \sqsubseteq N_2 : d}{\vdash_{d_\sigma} \text{let } x_1 = M_1 \text{ in } N_1 \sqsubseteq \text{let } x_2 = M_2 \text{ in } N_2 : d} \text{LETCONG}$$

$$\frac{\vdash_{d_\sigma} M \sqsubseteq M' : \text{bool} \quad \vdash_{d_\sigma} N_t \sqsubseteq N'_t : c \quad \vdash_{d_\sigma} N_f \sqsubseteq N'_f : c}{\vdash_{d_\sigma} \text{if } M\{N_t\}\{N_f\} \sqsubseteq \text{if } M'\{N'_t\}\{N'_f\} : c} \text{IFCONG}$$

$$\frac{c : A_1 \sqsubseteq A_2 \quad d : B_1 \sqsubseteq B_2 \quad \varepsilon@c \rightsquigarrow d \in d_\sigma \quad \vdash_{d_\sigma} M_1 \sqsubseteq M_2 : c}{\vdash_{d_\sigma} \text{raise } \varepsilon(M_1) \sqsubseteq \text{raise } \varepsilon(M_2) : d} \text{RAISECONG}$$

$$\frac{\vdash_{d_\sigma} M \sqsubseteq M' : c \quad y : c \vdash_{d_\tau} N \sqsubseteq N' : d \quad \forall \varepsilon@d_i \rightsquigarrow d_o \in d_\sigma. (\varepsilon \notin \text{dom}(\phi) \wedge \varepsilon \notin \text{dom}(\phi') \wedge \varepsilon : d_i \rightsquigarrow d_o \in d_\tau) \vee x : d_i, k : d_o \rightarrow_{d_\tau} d \vdash_{d_\tau} \phi(\varepsilon) \sqsubseteq \phi'(\varepsilon) : d}{\vdash_{d_\tau} \text{handle } M \{ \text{ret } y.N \mid \phi \} \sqsubseteq \text{handle } M' \{ \text{ret } y.N' \mid \phi' \} : d} \text{HANDLECONG}$$

Next, the rules for errors

$$\frac{\vdash_{d_\sigma} M : c^r}{\vdash_{d_\sigma} \mathbb{U} \sqsubseteq M : c} \text{ERRBOT} \quad E[\mathbb{U}] \equiv \mathbb{U} \text{ ERRSTRICT}$$

The generic rules for casts

$$\begin{array}{c}
\frac{\vdash_{d_\sigma} M \sqsubseteq N : (c : A \sqsubseteq B) \quad c : A \sqsubseteq A}{\vdash_{d_\sigma} \langle B \hookrightarrow A \rangle M \sqsubseteq N : B} \text{VALUP L} \qquad \frac{\vdash_\sigma M : A \quad c : A \sqsubseteq B}{\vdash_\sigma M \sqsubseteq \langle B \hookrightarrow A \rangle M : c} \text{VALUP R} \\
\langle B \hookrightarrow A \rangle M \equiv \text{let } x = M \text{ in } \langle B \hookrightarrow A \rangle x \quad \text{VALUPEVAL} \qquad \frac{c : A \sqsubseteq B \quad \vdash_\sigma N : B}{\vdash_\sigma \langle A \leftarrow B \rangle N \sqsubseteq N : c} \text{VALDNL} \\
\frac{\vdash_{d_\sigma} M \sqsubseteq N : (c : A \sqsubseteq B)}{\vdash_{d_\sigma} M \sqsubseteq \langle A \leftarrow B \rangle N : A} \text{VALDNR} \qquad \langle A \leftarrow B \rangle M \equiv \text{let } x = M \text{ in } \langle A \leftarrow B \rangle x \text{ VALDNEVAL} \\
\frac{\vdash_{d_\sigma} M \sqsubseteq N : c \quad d_\sigma : \sigma \sqsubseteq \tau}{\vdash_\tau \langle \tau \hookrightarrow \sigma \rangle M \sqsubseteq N : c} \text{VALUPL} \qquad \frac{\vdash_\sigma M : A \quad d_\sigma : \sigma \sqsubseteq \tau}{\vdash_{d_\sigma} M \sqsubseteq \langle \tau \hookrightarrow \sigma \rangle M : c} \text{VALUPR} \\
\frac{d_\sigma : \sigma \sqsubseteq \tau \quad \vdash_\tau N : A}{\vdash_{d_\sigma} \langle \sigma \leftarrow \tau \rangle N \sqsubseteq N : A} \text{EFFDNL} \qquad \frac{d_\sigma : \sigma \sqsubseteq \tau \quad \vdash_{d_\sigma} M \sqsubseteq N : c}{\vdash_\sigma M \sqsubseteq \langle \sigma \leftarrow \tau \rangle N : c} \text{EFFDNR}
\end{array}$$

And the subtyping rules

$$\begin{array}{c}
\frac{\vdash_{d_\sigma} M \sqsubseteq N : c \quad d_\sigma : \sigma \sqsubseteq \tau \quad c : A \sqsubseteq B \quad d'_\sigma : \sigma' \sqsubseteq \tau' \quad c' : A' \sqsubseteq B' \quad \sigma \leq : \sigma' \quad A \leq : A' \quad \tau \leq : \tau' \quad B \leq : B'}{\vdash_{d'_\sigma} M \sqsubseteq N : c'} \text{SUBTYMON} \\
\frac{c : A \sqsubseteq B \quad c' : A' \sqsubseteq B' \quad c \leq : c' \quad \vdash_\sigma M : A}{\vdash_\sigma \langle B \hookrightarrow A \rangle M \equiv \langle B' \hookrightarrow A' \rangle M : B'} \text{VALUPSUB} \\
\frac{c : A \sqsubseteq B \quad c' : A' \sqsubseteq B' \quad c \leq : c' \quad \vdash_\sigma N : B}{\vdash_\sigma \langle A \leftarrow B \rangle N \equiv \langle A' \leftarrow B' \rangle N : \sigma ! A'} \text{VALDNSUB} \\
\frac{c_\sigma : \sigma \sqsubseteq \tau \quad c' : \sigma' \sqsubseteq \tau' \quad c_\sigma \leq : c'_\sigma \quad \vdash_\sigma M : A}{\vdash_{\tau'} \langle \tau \hookrightarrow \sigma \rangle M \equiv \langle \tau' \hookrightarrow \sigma' \rangle M : A} \text{EFFUPSUB} \\
\frac{c_\sigma : \sigma \sqsubseteq \tau \quad c' : \sigma' \sqsubseteq \tau' \quad c_\sigma \leq : c'_\sigma \quad \vdash_\tau N : A}{\vdash_{\sigma'} \langle \sigma \leftarrow \tau \rangle N \equiv \langle \sigma' \leftarrow \tau' \rangle N : A} \text{EFFDNSUB}
\end{array}$$

In Figure 17, we list some derivable reasoning principles for our inequational theory, which can be used to derive the operational semantics.

We can show the following properties of the interaction between subtyping and casts axiomatically:

LEMMA B.1. *The following hold:*

- (1) $\Sigma \mid \Gamma \sqsubseteq \vDash_{d_\sigma} \langle B' \hookrightarrow A' \rangle M \sqsubseteq \langle B \hookrightarrow A \rangle N : B'$.
- (2) $\Sigma \mid \Gamma \sqsubseteq \vDash_{d_\sigma} \langle A \leftarrow B \rangle M \sqsubseteq \langle A' \leftarrow B' \rangle N : A'$.
- (3) $\Sigma \mid \Gamma \sqsubseteq \vDash_{\sigma'_2} \langle \sigma'_2 \hookrightarrow \sigma'_1 \rangle P \sqsubseteq \langle \sigma_2 \hookrightarrow \sigma_1 \rangle Q : c$.
- (4) $\Sigma \mid \Gamma \sqsubseteq \vDash_{\sigma'_1} \langle \sigma_1 \leftarrow \sigma_2 \rangle P \sqsubseteq \langle \sigma'_1 \leftarrow \sigma'_2 \rangle Q : c$.

PROOF.

$$\begin{array}{c}
\langle A \frown A \rangle M \equiv M \quad \langle \sigma \frown \sigma \rangle M \equiv M \quad \langle A \ll A \rangle M \equiv M \quad \langle \sigma \ll \sigma \rangle M \equiv M \\
\langle C \frown B \rangle \langle B \frown A \rangle M \equiv \langle C \frown A \rangle M \quad \langle A \ll B \rangle \langle B \ll C \rangle M \equiv \langle A \ll C \rangle M \\
\langle \sigma'' \frown \sigma' \rangle \langle \sigma' \frown \sigma \rangle M \equiv \langle \sigma'' \frown \sigma \rangle M \quad \langle \sigma \ll \sigma' \rangle \langle \sigma' \ll \sigma'' \rangle M \equiv \langle \sigma \ll \sigma'' \rangle M \\
\langle B \frown A \rangle \langle \sigma' \frown \sigma \rangle M \equiv \langle \sigma' \frown \sigma \rangle \langle B \frown A \rangle M \quad \langle A \ll B \rangle \langle \sigma \ll \sigma' \rangle M \equiv \langle \sigma \ll \sigma' \rangle \langle A \ll B \rangle M \\
\langle A' \rightarrow_{\sigma'} B' \frown A \rightarrow_{\sigma} B \rangle V \equiv \lambda x'. \langle B' \frown B \rangle \langle \sigma' \frown \sigma \rangle (V(\langle A \ll A' \rangle x')) \\
\langle A \rightarrow_{\sigma} B \ll A' \rightarrow_{\sigma'} B' \rangle V \equiv \lambda x. \langle B \ll B' \rangle \langle \sigma \ll \sigma' \rangle (V(\langle A' \frown A \rangle x)) \\
\frac{\text{dom}(\phi) = \text{dom}(\sigma) \quad \forall \varepsilon \in \sigma. \phi(\varepsilon) = k(\text{raise } \varepsilon(x))}{\Sigma \mid \Gamma \vdash \langle \tau \frown \sigma \rangle M \equiv \text{handle } M \{ \text{ret } x.x \mid \phi_{\langle \tau \frown \sigma \rangle} \}} \\
\frac{\text{dom}(\phi) = \text{dom}(\tau) \quad \forall \varepsilon \in \tau. (\varepsilon \in \sigma \implies \phi(\varepsilon) = k(\text{raise } \varepsilon(x))) \wedge (\varepsilon \notin \sigma \implies \phi(\varepsilon) = \mathcal{U})}{\Sigma \mid \Gamma \vdash \langle \sigma \ll \tau \rangle M \equiv \text{handle } M \{ \text{ret } x.x \mid \phi_{\langle \tau \frown \sigma \rangle} \}}
\end{array}$$

Fig. 17. Proveable Uniqueness Theorems

We have

$$\frac{\frac{\frac{\vdash_{d\sigma} M \sqsubseteq N : A}{\vdash_{d\sigma} M \sqsubseteq \langle B \frown A \rangle N : A \sqsubseteq B} \text{ (VALUPR)}}{\vdash_{d\sigma} M \sqsubseteq \langle B \frown A \rangle N : A' \sqsubseteq B'} \text{ (SUBTYPING)}}{\vdash_{d\sigma} \langle B' \frown A' \rangle M \sqsubseteq \langle B \frown A \rangle N : B'} \text{ (VALUPL)}$$

Dual to the above.

We have

$$\frac{\frac{\frac{\vdash_{\sigma_1} P \sqsubseteq Q : c}{\vdash_{d\sigma} P \sqsubseteq \langle \sigma_2 \frown \sigma_1 \rangle Q : c} \text{ (EFFUPR)}}{\vdash_{d\sigma} P \sqsubseteq \langle \sigma_2 \frown \sigma_1 \rangle Q : c} \text{ (SUBTYPING)}}{\vdash_{\sigma'_2} \langle \sigma'_2 \frown \sigma'_1 \rangle P \sqsubseteq \langle \sigma_2 \frown \sigma_1 \rangle Q : c} \text{ (EFFUPL)}$$

Dual to the above. □

C OPERATIONAL SEMANTICS

An evaluation context $E_{\# \epsilon}$ is one in which none of the handler clauses in the spine of the context handles ϵ .

C.1 Operational Semantics from First Principles

Now we show that every operational reduction is justified by our inequational theory.

LEMMA C.1 (EFFECT CASTS ARE HANDLERS). *Let $\sigma \sqsubseteq \tau$ where σ is a concrete effect set.*

$$\begin{array}{c}
\frac{\varepsilon \in \text{dom}(\phi) \quad E' \# \varepsilon}{E[\text{handle } E'[\text{raise } (\varepsilon @ A \rightsquigarrow B)(V)] \{ \text{ret } x.N \mid \phi \}] \mapsto E[\phi(\varepsilon)[V/x][(\lambda y. \text{handle } (E'[y]) \{ \text{ret } x.N \mid \phi \})/k]]} \\
\text{HANDLEVAL} \\
\frac{}{E[\text{handle } V \{ \text{ret } x.N \mid \phi \}] \mapsto E[N[V/x]]} \\
\frac{}{E[(\lambda x.M)V] \mapsto E[M[V/x]]} \text{LAM} \quad \frac{}{E[\text{let } x = V \text{ in } M] \mapsto E[M[V/x]]} \text{LET} \\
\frac{}{E[\mathbb{U}] \mapsto \mathbb{U}} \text{ERR} \quad \frac{}{E[\text{if true}\{N_t\}\{N_f\}] \mapsto E[N_t]} \text{IFTRUE} \\
\frac{}{E[\text{if false}\{N_t\}\{N_f\}] \mapsto E[N_f]} \text{IFFALSE} \quad \frac{}{E[\langle \sigma' \rightsquigarrow \sigma \rangle V] \mapsto E[V]} \text{EFFUPDNCSTVAL} \\
\frac{}{E[\langle \sigma \leftarrow \sigma' \rangle V] \mapsto E[V]} \text{EFFDNCSTVAL} \\
\frac{\varepsilon @ A \rightsquigarrow B' \in \sigma' \quad E' \# \varepsilon}{E[\langle \sigma' \rightsquigarrow \sigma \rangle E'[\text{raise } (\varepsilon @ A \rightsquigarrow B)(V)]] \mapsto E[\text{let } x = \langle B \leftarrow B' \rangle \text{raise } \varepsilon(\langle A' \rightsquigarrow A \rangle V) \text{ in } \langle \sigma' \rightsquigarrow \sigma \rangle E'[x]]} \text{EFFUPCAST} \\
\frac{\varepsilon @ A \rightsquigarrow B \in \sigma \quad E' \# \varepsilon}{E[\langle \sigma \leftarrow \sigma' \rangle E'[\text{raise } (\varepsilon @ A' \rightsquigarrow B')(V)]] \mapsto E[\text{let } x = \langle B' \rightsquigarrow B \rangle \text{raise } \varepsilon(\langle A \leftarrow A' \rangle V) \text{ in } \langle \sigma \leftarrow \sigma' \rangle E'[x]]} \text{GOODEFFDNCST} \\
\frac{\varepsilon @ A \rightsquigarrow B \notin \sigma \quad E' \# \varepsilon}{E[\langle \sigma \leftarrow ? \rangle E'[\text{raise } (\varepsilon @ A' \rightsquigarrow B')(V)]] \mapsto E[\mathbb{U}]} \text{BADEFFDNCST} \\
E[\uparrow \text{bool}M] \mapsto E[M] \text{ BOOLUPDNCST} \\
\frac{}{E[\langle \langle (A' \rightarrow_{\sigma'} B') \rightsquigarrow (A \rightarrow_{\sigma} B) \rangle V_f \rangle V] \mapsto E[\langle B' \rightsquigarrow B \rangle \langle \sigma' \rightsquigarrow \sigma \rangle \langle V_f \langle A \leftarrow A' \rangle V \rangle]} \text{FUNUPCAST} \\
\frac{}{E[\langle \langle (A \rightarrow_{\sigma} B) \leftarrow (A' \rightarrow_{\sigma'} B') \rangle V_f \rangle V] \mapsto E[\langle B \leftarrow B' \rangle \langle \sigma \leftarrow \sigma' \rangle \langle V_f \langle A' \rightsquigarrow A \rangle V \rangle]} \text{FUNDNCST}
\end{array}$$

Fig. 18. Full Operational Semantics

Then the upcast $\langle \tau \rightsquigarrow \sigma \rangle$ is equivalent to a handler in that for any $M : \sigma \uparrow A$:

$$\langle \tau \rightsquigarrow \sigma \rangle M \equiv \text{handle } M \{ \text{ret } x.x \mid \phi_{\langle \tau \rightsquigarrow \sigma \rangle} \}$$

where for each $\varepsilon \in \text{dom}(\sigma)$

$$x, k \vdash \phi_{\langle \tau \rightsquigarrow \sigma \rangle}(\varepsilon) = k(\langle B_{\sigma} \leftarrow B_{\tau} \rangle \text{raise } \varepsilon(\langle A_{\tau} \rightsquigarrow A_{\sigma} \rangle))$$

where $\varepsilon @ A_{\sigma} \rightsquigarrow B_{\sigma} \in \sigma$ and $\varepsilon @ A_{\tau} \rightsquigarrow B_{\tau} \in \tau$.

$$\begin{array}{c}
\Delta ::= \bullet : (\sigma!A) \quad \frac{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} E : A \rightarrow_{\sigma} B \quad \Sigma \mid \Gamma \mid \cdot \vdash_{\sigma} N : A}{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} EN : B} \\
\\
\frac{\Sigma \mid \Gamma \vdash_{\sigma} V : A \rightarrow_{\sigma} B \quad \Sigma \mid \Gamma \mid \bullet : (\sigma_i!C) \vdash_{\sigma} E : A}{\Sigma \mid \Gamma \mid \bullet : (\sigma_i!C) \vdash_{\sigma} VE : B} \\
\\
\frac{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} E : \text{bool} \quad \Sigma \mid \Gamma \vdash_{\sigma} N_t B \quad \Sigma \mid \Gamma \vdash_{\sigma} N_f B}{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} \text{if } E\{N_t\}\{N_f\} : B} \\
\\
\frac{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} E : A \quad \epsilon @ A \rightsquigarrow B \in \sigma}{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} \text{raise } (\epsilon @ A \rightsquigarrow B)(E) : B} \\
\\
\frac{\begin{array}{c} \Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} E : A \\ \Sigma \mid \Gamma, x : A \vdash_{\sigma_0} N : C \\ (\forall i \in I. \epsilon_i @ A_i \rightsquigarrow B_i \in \sigma \wedge \Sigma \mid \Gamma, x_i : A_i, k_i : B_i \rightarrow_{\sigma_0} C \vdash_{\sigma_0} N_i : C) \\ \sigma - \{\epsilon_i @ A_i \rightsquigarrow B_i\}_{i \in I} \leq: \sigma_0 \end{array}}{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma_0} \text{handle } E \{ \text{ret } x.N \mid (\epsilon_i @ A_i \rightsquigarrow B_i)(x_i, k_i) \mapsto N_i \dots_{i \in I} \} : C} \\
\\
\frac{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} E : A \quad \Sigma \mid \Gamma, x : A \vdash_{\sigma} N : B}{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} \text{let } x = E \text{ in } N : B} \quad \frac{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} E : A' \quad \Sigma \mid \Gamma \vdash A' \leq: A}{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} E : A} \\
\\
\frac{\Sigma \mid \Gamma \mid \Delta \vdash_{\Delta} \sigma' : EA \quad \Sigma \mid \Gamma \vdash \sigma' \leq: \sigma}{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} E : A} \quad \frac{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} E : A \quad \Sigma \vdash A \sqsubseteq B}{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} \langle B \prec_{\sigma} A \rangle E : B} \\
\\
\frac{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} E : B \quad \Sigma \vdash A \sqsubseteq B}{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} \langle A \preccurlyeq B \rangle E : A} \quad \frac{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} E : A \quad \Sigma \vdash \sigma \sqsubseteq \sigma'}{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma'} \langle \sigma' \prec_{\sigma} \sigma \rangle E : A} \\
\\
\frac{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma'} E : A \quad \Sigma \vdash \sigma \sqsubseteq \sigma'}{\Sigma \mid \Gamma \mid \Delta \vdash_{\sigma} \langle \sigma \preccurlyeq \sigma' \rangle E : A}
\end{array}$$

Fig. 19. Typing Rules for Evaluation Contexts

$$\begin{array}{c}
\epsilon \# \bullet \quad \frac{\epsilon \# E}{\epsilon \# (\langle B \prec_{\sigma} A \rangle E)} \quad \frac{\epsilon \# E}{\epsilon \# (\langle A \preccurlyeq B \rangle E)} \quad \frac{\epsilon \# E \quad \epsilon \notin \sigma \quad \epsilon \notin \sigma'}{\epsilon \# (\langle \sigma' \prec_{\sigma} \sigma \rangle E)} \quad \frac{\epsilon \# E \quad \epsilon \notin \sigma'}{\epsilon \# (\langle \sigma \preccurlyeq \sigma' \rangle E)} \\
\\
\frac{\epsilon \# E \quad \epsilon' \text{ any effect}}{\epsilon \# (\text{raise } (\epsilon' @ A \rightsquigarrow B)(E))} \quad \frac{\epsilon \# E \wedge \forall i \in I. \epsilon_i \neq \epsilon}{\epsilon \# (\text{handle } E \{ \text{ret } x.N \mid (\epsilon_i @ A_i \rightsquigarrow B_i)(x_i, k_i) \mapsto N_i \dots_{i \in I} \})} \\
\\
\frac{\epsilon \# E}{\epsilon \# (EM)} \quad \frac{\epsilon \# E}{\epsilon \# (VE)} \quad \frac{\epsilon \# E}{\epsilon \# (\text{if } E\{N_t\}\{N_f\})} \quad \frac{\epsilon \# E}{\epsilon \# (\text{let } x = E \text{ in } N)}
\end{array}$$

Fig. 20. Apartness of Effect from an Evaluation Context

Similarly, the downcast $\langle \sigma \leftarrow \tau \rangle$ is equivalent to a handler in that for any $N : \tau ! A$:

$$\langle \sigma \leftarrow \tau \rangle M \equiv \text{handle } M \{ \text{ret } x.x \mid \phi_{\langle \sigma \leftarrow \tau \rangle} \}$$

where for each $\varepsilon \in \text{dom}(\tau)$, if $\varepsilon \in \text{dom}(\sigma)$, then

$$x, k \vdash \phi_{\langle \sigma \leftarrow \tau \rangle}(\varepsilon) = k(\langle B_\tau \leftarrow B_\sigma \rangle \text{raise } \varepsilon(\langle A_\sigma \leftarrow A_\tau \rangle))$$

and if $\varepsilon \notin \text{dom}(\sigma)$, then

$$\phi_{\langle \sigma \leftarrow \tau \rangle}(\varepsilon) = \mathbf{U}$$

PROOF. First for the upcast case

- We want to show

$$\langle \tau \leftarrow \sigma \rangle M \sqsubseteq \text{handle } M \{ \text{ret } x.x \mid \phi_{\langle \tau \leftarrow \sigma \rangle} \}$$

By UpL, it is sufficient to show

$$M \sqsubseteq \text{handle } M \{ \text{ret } x.x \mid \phi_{\langle \tau \leftarrow \sigma \rangle} \}$$

But by the handler η rule, this is equivalent to showing

$$\text{handle } M \{ \text{ret } x.x \mid \phi_\sigma \} \sqsubseteq \text{handle } M \{ \text{ret } x.x \mid \phi_{\langle \tau \leftarrow \sigma \rangle} \}$$

where $\text{dom}(\phi_\sigma) = \text{dom}(\sigma)$ and $\phi_\sigma(\varepsilon) = k(\text{raise } \varepsilon(x))$. Then by congruence, we need to show that for each $\varepsilon \in \text{dom}(\sigma)$,

$$k(\text{raise } \varepsilon(x)) \sqsubseteq k(\langle B_\sigma \leftarrow B_\tau \rangle \text{raise } \varepsilon(\langle A_\sigma \leftarrow A_\tau \rangle))$$

which follows from UpR/DnR and congruence rules

- We want to show

$$\text{handle } M \{ \text{ret } x.x \mid \phi_{\langle \tau \leftarrow \sigma \rangle} \} \sqsubseteq \langle \tau \leftarrow \sigma \rangle M$$

By handler η it is sufficient to show

$$\text{handle } M \{ \text{ret } x.x \mid \phi_{\langle \tau \leftarrow \sigma \rangle} \} \sqsubseteq \text{handle } \langle \tau \leftarrow \sigma \rangle M \{ \text{ret } x.x \mid \phi_\tau \}$$

where $\text{dom}(\phi_\tau) = \text{dom}(\tau)$ and $\phi_\tau(\varepsilon) = k(\text{raise } \varepsilon(x))$. Then $M \sqsubseteq \langle \tau \leftarrow \sigma \rangle M$ by UpR and so by congruence we need only to show for each $\varepsilon \in \sigma$ that

$$\phi_{\langle \tau \leftarrow \sigma \rangle}(\varepsilon) \sqsubseteq \phi_\tau(\varepsilon)$$

which follows by a similar argument to the previous case.

Next, the downcast cases.

- We want to show

$$\text{handle } N \{ \text{ret } x.x \mid \phi_{\langle \sigma \leftarrow \tau \rangle} \} \sqsubseteq \langle \sigma \leftarrow \tau \rangle N$$

By DnR, it is sufficient to show

$$\text{handle } N \{ \text{ret } x.x \mid \phi_{\langle \sigma \leftarrow \tau \rangle} \} \sqsubseteq N$$

By handler η this is equivalent to showing

$$\text{handle } N \{ \text{ret } x.x \mid \phi_{\langle \sigma \leftarrow \tau \rangle} \} \sqsubseteq \text{handle } N \{ \text{ret } x.x \mid \phi_\tau \}$$

That is, for any $\varepsilon \in \text{dom}(\tau)$ that

$$\phi_{\langle \sigma \leftarrow \tau \rangle}(\varepsilon) \sqsubseteq \phi_\tau(\varepsilon)$$

There are two cases

(1) If $\varepsilon \in \text{dom}(\sigma)$, then we need to show

$$k(\langle B_\tau \frown B_\sigma \rangle \text{raise } \varepsilon(\langle A_\tau \frown A_\sigma \rangle x)) \sqsubseteq k(\text{raise } \varepsilon(x))$$

which follows by congruence and DnL/UpL rules.

(2) If $\varepsilon \notin \text{dom}(\sigma)$, then we need to show

$$\top \sqsubseteq k(\text{raise } \varepsilon(x))$$

which is immediate.

• We want to show

$$\langle \sigma \ll \tau \rangle N \sqsubseteq \text{handle } N \{ \text{ret } x.x \mid \phi_{\langle \sigma \ll \tau \rangle} \}$$

By handler η this is equivalent to showing

$$\text{handle } (\langle \sigma \ll \tau \rangle N) \{ \text{ret } x.x \mid \phi_\sigma \} \sqsubseteq \text{handle } N \{ \text{ret } x.x \mid \phi_{\langle \sigma \ll \tau \rangle} \}$$

By congruence and DnL this reduces to showing for each $\varepsilon \in \text{dom}(\sigma)$ that

$$\phi_\sigma(\varepsilon) \sqsubseteq \phi_{\langle \sigma \ll \tau \rangle}(\varepsilon)$$

since $\varepsilon \in \text{dom}(\sigma)$, these are each of the form:

$$k(\text{raise } \varepsilon(x)) \sqsubseteq k(\langle B_\tau \frown B_\sigma \rangle \text{raise } \varepsilon(\langle A_\tau \frown A_\sigma \rangle x))$$

which follows by congruence and DnR/UpR rules. □

LEMMA C.2 (DERIVATION OF FUNCTION CASTS).

$$\langle A' \rightarrow_\tau B' \frown A \rightarrow_\sigma B \rangle f \equiv \lambda x. \langle B' \frown B \rangle \langle \tau \frown \sigma \rangle (f(\langle A \ll A' \rangle x))$$

And similarly,

$$\langle A \rightarrow_\sigma B \ll A' \rightarrow_\tau B' \rangle f \equiv \lambda x. \langle B \ll B' \rangle \langle \sigma \ll \tau \rangle (f(\langle A' \frown A \rangle x))$$

PROOF. We show the upcast cases, the downcast cases are precisely dual.

(1) We want to show

$$\langle A' \rightarrow_\tau B' \frown A \rightarrow_\sigma B \rangle f \sqsubseteq \lambda x. \langle B' \frown B \rangle \langle \tau \frown \sigma \rangle (f(\langle A \ll A' \rangle x))$$

By UpL, it is sufficient to show

$$f \sqsubseteq \lambda x. \langle B' \frown B \rangle \langle \tau \frown \sigma \rangle (f(\langle A \ll A' \rangle x))$$

By η equivalence for functions it is sufficient to show

$$\lambda x. f x \sqsubseteq \lambda x. \langle B' \frown B \rangle \langle \tau \frown \sigma \rangle (f(\langle A \ll A' \rangle x))$$

Which follows by congruence rules and UpR/DnR rules.

(2) We want to show

$$\lambda x. \langle B' \frown B \rangle \langle \tau \frown \sigma \rangle (f(\langle A \ll A' \rangle x)) \sqsubseteq \langle A' \rightarrow_\tau B' \frown A \rightarrow_\sigma B \rangle f$$

By function η it is sufficient to show

$$\lambda x. \langle B' \frown B \rangle \langle \tau \frown \sigma \rangle (f(\langle A \ll A' \rangle x)) \sqsubseteq \lambda y. (\langle A' \rightarrow_\tau B' \frown A \rightarrow_\sigma B \rangle f) y$$

Which follows by congruence and UpL/DnL/UpR rules. □

LEMMA C.3. If $x, k \vdash \phi(\varepsilon) = k(\text{raise } \varepsilon(x))$, then

$$\text{handle raise } \varepsilon(x) \{ \text{ret } y.N \mid \phi \} \equiv \text{let } o = (\text{raise } \varepsilon(x)) \text{ in handle } o \{ \text{ret } y.N \mid \phi \}$$

PROOF.

$$\begin{aligned} \text{handle raise } \varepsilon(x) \{ \text{ret } y.N \mid \phi \} &\equiv \text{handle } (\text{let } o = \text{raise } \varepsilon(x) \text{ in } o \{ \text{ret } y.N \mid \phi \}) \\ &\equiv (\lambda o. \text{handle } o \{ \text{ret } y.N \mid \phi \}) (\text{raise } \varepsilon(x)) \\ &\equiv \text{let } o = (\text{raise } \varepsilon(x)) \text{ in handle } o \{ \text{ret } y.N \mid \phi \} \end{aligned}$$

□

This lemma is useful for the cast cases of the following, as it reduces to showing the cast is equivalent to one whose ε case is just a re-raise.

LEMMA C.4. *If $E \# \varepsilon$, then*

$$E[\text{raise } \varepsilon(x)] \equiv \text{let } y = \text{raise } \varepsilon(x) \text{ in } E[y]$$

PROOF. By induction on $\varepsilon \# E$

- $\varepsilon \# \bullet$

$$\text{raise } \varepsilon(x) \equiv \text{let } y = \text{raise } \varepsilon(x) \text{ in } y$$

- $\frac{\varepsilon \# E}{\varepsilon \# (\langle B \prec_A A \rangle E)}$

$$\begin{aligned} \langle B \prec_A A \rangle E[\text{raise } \varepsilon(x)] &\equiv \text{let } [= E \text{ in raise } \varepsilon(x)] y \langle B \prec_A A \rangle y \\ &\equiv \text{let } y = (\text{let } z = (\text{raise } \varepsilon(x)) \text{ in } E[z]) \text{ in } \langle B \prec_A A \rangle y \\ &\equiv \text{let } z = (\text{raise } \varepsilon(x)) \text{ in let } y = E[z] \text{ in } \langle B \prec_A A \rangle y \\ &\equiv \text{let } z = (\text{raise } \varepsilon(x)) \text{ in } \langle B \prec_A A \rangle E[z] \end{aligned}$$

- $\frac{\varepsilon \# E}{\varepsilon \# (\langle A \leftarrow B \rangle E)}$
Similar to previous.

- $\frac{\varepsilon \# E}{\varepsilon \# (\text{raise } \varepsilon'(E))}$

$$\begin{aligned} \text{raise } \varepsilon'(E[\text{raise } \varepsilon'(x)]) &\equiv \text{raise } \varepsilon'((\text{let } z = \text{raise } \varepsilon'(x) \text{ in } E[z])) \\ &\equiv \text{let } z = \text{raise } \varepsilon'(x) \text{ in } E[z] \text{raise } \varepsilon'(\text{ }) \end{aligned}$$

- $\frac{\varepsilon \# E \quad \varepsilon \notin \text{dom}(\phi)}{\varepsilon \# (\text{handle } E \{ \text{ret } y.N \mid \phi \})}$

Define ψ to be the extension of ϕ with the case $\psi(\varepsilon) = k(\text{raise } \varepsilon(x))$.

$$\begin{aligned} \text{handle } E[\text{raise } \varepsilon(x)] \{ \text{ret } y.N \mid \phi \} &\equiv \text{handle } E[\text{raise } \varepsilon(x)] \{ \text{ret } y.N \mid \psi \} \\ &\equiv \text{handle } (\text{let } z = (\text{raise } \varepsilon(x)) \text{ in } E[z]) \{ \text{ret } y.N \mid \psi \} \\ &\equiv (\lambda o. \text{handle } E[o] \{ \text{ret } y.N \mid \psi \}) (\text{raise } \varepsilon(x)) \\ &\equiv (\text{let } o = (\text{raise } \varepsilon(x)) \text{ in handle } E[o] \{ \text{ret } y.N \mid \psi \}) \\ &\equiv (\text{let } o = (\text{raise } \varepsilon(x)) \text{ in handle } E[o] \{ \text{ret } y.N \mid \phi \}) \end{aligned}$$

$$\bullet \frac{\epsilon \# E}{\epsilon \# (EM)}$$

$$\begin{aligned} (E[\text{raise } \epsilon(x)])M &\equiv \text{let } f = E[\text{raise } \epsilon(x)] \text{ in let } y = M \text{ in } f y \\ &\equiv \text{let } f = (\text{let } z = \text{raise } \epsilon(x) \text{ in } E[z]) \text{ in let } y = M \text{ in } f y \\ &\equiv \text{let } z = \text{raise } \epsilon(x) \text{ in let } f = E[z] \text{ in let } y = M \text{ in } f y \\ &\equiv \text{let } z = \text{raise } \epsilon(x) \text{ in } (E[z]) M \end{aligned}$$

$$\bullet \frac{\epsilon \# E}{\epsilon \# (VE)}$$

$$\begin{aligned} (VE[\text{raise } \epsilon(x)]) &\equiv \text{let } f = V \text{ in let } y = E[\text{raise } \epsilon(x)] \text{ in } f y \\ &\equiv \text{let } f = V \text{ in let } y = (\text{let } z = \text{raise } \epsilon(x) \text{ in } E[z]) \text{ in } f y \\ &\equiv \text{let } y = (\text{let } z = \text{raise } \epsilon(x) \text{ in } E[z]) \text{ in } V y \\ &\equiv \text{let } z = \text{raise } \epsilon(x) \text{ in let } y = (E[z]) \text{ in } V y \\ &\equiv \text{let } z = \text{raise } \epsilon(x) \text{ in } V (E[z]) \end{aligned}$$

$$\bullet \frac{\epsilon \# E}{\epsilon \# (\text{if } E\{N_t\}\{N_f\})}$$

$$\begin{aligned} \text{if } E[\text{raise } \epsilon(x)]\{N_t\}\{N_f\} &\equiv \text{let } y = (E[\text{raise } \epsilon(x)]) \text{ in if } y\{N_t\}\{N_f\} \\ &\equiv \text{let } y = (\text{let } z = \text{raise } \epsilon(x) \text{ in } E[z]) \text{ in if } y\{N_t\}\{N_f\} \\ &\equiv \text{let } z = \text{raise } \epsilon(x) \text{ in let } y = (E[z]) \text{ in if } y\{N_t\}\{N_f\} \\ &\equiv \text{let } z = \text{raise } \epsilon(x) \text{ in if } E[z]\{N_t\}\{N_f\} \end{aligned}$$

$$\bullet \frac{\epsilon \# E}{\epsilon \# (\text{let } x = E \text{ in } N)}$$

$$\begin{aligned} \text{let } y = E[\text{raise } \epsilon(x)] \text{ in } N &\equiv \text{let } y = \text{let } z = (\text{raise } \epsilon(x)) \text{ in } E[z] \text{ in } N \\ &\equiv \text{let } z = (\text{raise } \epsilon(x)) \text{ in let } y = E[z] \text{ in } N \end{aligned}$$

□

THEOREM C.5 (SOUNDNESS OF OPERATIONAL SEMANTICS). *If $M \mapsto^* M'$ then $M \equiv M'$ is derivable in the inequational theory.*

PROOF. (1) The value handle, boolean/function β reductions and error reduction are immediate by axioms.

(2)

$$\frac{E \# \epsilon}{\text{handle } E[\text{raise } \epsilon(V)] \{ \text{ret } y.N \mid \phi \} \equiv \phi(\epsilon)[V/x, \lambda o. \text{handle } E[o] \{ \text{ret } y.N \mid \phi \} / k]}$$

$$\begin{aligned} \text{handle } E[\text{raise } \varepsilon(V)] \{ \text{ret } y.N \mid \phi \} &\equiv \text{handle } (\text{let } z = \text{raise } \varepsilon(V) \text{ in } E[z]) \{ \text{ret } y.N \mid \phi \} \\ &\quad \text{(Lemma C.4)} \\ &\equiv \phi(\varepsilon)[V/x, \lambda o. \text{handle } E[o] \{ \text{ret } y.N \mid \phi \} / k] \end{aligned}$$

(3)

$$\langle \tau \prec \sigma \rangle V \equiv V$$

By the following:

$$\begin{aligned} \langle \tau \prec \sigma \rangle V &\equiv \text{handle } V \{ \text{ret } x.x \mid \phi_{\langle \tau \prec \sigma \rangle} \} && \text{(Lemma C.1)} \\ &\equiv V && \text{(Handle } \beta) \end{aligned}$$

(4)

$$\langle \sigma \prec \tau \rangle V \equiv V$$

is similar to the previous.

(5)

$$\frac{\varepsilon @ A \rightsquigarrow B \in \sigma \quad \varepsilon @ A' \rightsquigarrow B' \in \tau \quad E \# \varepsilon}{\langle \tau \prec \sigma \rangle E[\text{raise } \varepsilon(V)] \equiv \text{let } x = \langle B \prec B' \rangle \text{raise } \varepsilon(\langle A' \prec A \rangle V) \text{ in } \langle \tau \prec \sigma \rangle E[x]}$$

$$\begin{aligned} \langle \tau \prec \sigma \rangle E[\text{raise } \varepsilon(V)] &\equiv \text{handle } (E[\text{raise } \varepsilon(V)]) \{ \text{ret } x.x \mid \phi_{\langle \tau \prec \sigma \rangle} \} && \text{(Lemma C.1)} \\ &\equiv \text{handle } (\text{let } z = \text{raise } \varepsilon(V) \text{ in } E[z]) \{ \text{ret } x.x \mid \phi_{\langle \tau \prec \sigma \rangle} \} && \text{(Lemma C.4)} \\ &\equiv \phi_{\langle \tau \prec \sigma \rangle}(\varepsilon)[V/x, \lambda o. \text{handle } E[o] \{ \text{ret } x.x \mid \phi_{\langle \tau \prec \sigma \rangle} \}] \\ &= (\lambda o. \text{handle } E[o] \{ \text{ret } x.x \mid \phi_{\langle \tau \prec \sigma \rangle} \}) (\langle B \prec B' \rangle \text{raise } \varepsilon(\langle A' \prec A \rangle V)) \\ &\equiv (\lambda o. \langle \tau \prec \sigma \rangle E[o]) (\langle B \prec B' \rangle \text{raise } \varepsilon(\langle A' \prec A \rangle V)) \\ &\equiv \text{let } o = (\langle B \prec B' \rangle \text{raise } \varepsilon(\langle A' \prec A \rangle V)) \text{ in } \langle \tau \prec \sigma \rangle E[o] \end{aligned}$$

(6)

$$\frac{\varepsilon @ A \rightsquigarrow B \in \sigma \quad \varepsilon @ A' \rightsquigarrow B' \in \tau \quad E \# \varepsilon}{\langle \sigma \prec \tau \rangle E[\text{raise } \varepsilon(V)] \equiv \text{let } x = \langle B' \prec B \rangle \text{raise } \varepsilon(\langle A \prec A' \rangle V) \text{ in } \langle \sigma \prec \tau \rangle E[x]}$$

Similar to previous

(7)

$$\frac{\varepsilon \notin \sigma \quad E \# \varepsilon}{\langle \sigma \prec ? \rangle E[\text{raise } \varepsilon(V)] \equiv \mathcal{U}}$$

$$\begin{aligned} \langle \sigma \prec ? \rangle E[\text{raise } \varepsilon(V)] &\equiv \text{handle } (E[\text{raise } \varepsilon(V)]) \{ \text{ret } x.x \mid \phi_{\langle \sigma \prec ? \rangle} \} && \text{(Lemma C.1)} \\ &\equiv \text{handle } (\text{let } z = (\text{raise } \varepsilon(V)) \text{ in } E[z]) \{ \text{ret } x.x \mid \phi_{\langle \sigma \prec ? \rangle} \} && \text{(Lemma C.4)} \\ &\equiv \mathcal{U} \end{aligned}$$

(8)

$$\langle \text{bool} \prec \text{bool} \rangle V \equiv V$$

By the identity rule.

(9)

$$\langle \text{bool} \leftarrow \text{bool} \rangle V \equiv V$$

By the identity rule.

(10)

$$(\langle (A' \rightarrow_\tau B') \leftarrow (A \rightarrow_\sigma B) \rangle V_f) V \equiv \langle B' \leftarrow B \rangle \langle \tau \leftarrow \sigma \rangle (V_f \langle A \leftarrow A' \rangle V)$$

By the following:

$$\begin{aligned} (\langle (A' \rightarrow_\tau B') \leftarrow (A \rightarrow_\sigma B) \rangle V_f) V &\equiv ((\lambda x. \langle B' \leftarrow B \rangle \langle \tau \leftarrow \sigma \rangle (V_f \langle A \leftarrow A' \rangle x))) V && \text{(Lemma C.2)} \\ &\equiv \langle B' \leftarrow B \rangle \langle \tau \leftarrow \sigma \rangle (V_f \langle A \leftarrow A' \rangle V) && (\beta \rightarrow) \end{aligned}$$

(11) Similar to previous. □

THEOREM C.6 (ADEQUACY). *If $\vdash_\emptyset M \equiv M' : \text{bool}$ is derivable in the equational theory than for any $R \in \{\text{true}, \text{false}, \mathcal{U}\}$*

$$M \mapsto^* R \iff M' \mapsto^* R$$

COROLLARY C.7 (CONSISTENCY). *$\text{true} \equiv \text{false}$ is not derivable.*

THEOREM C.8 (GRADUALITY). *If $\vdash_\emptyset M \sqsubseteq M' : \text{bool}$ Then for any $R \in \{\text{true}, \text{false}\}$,*

$$M \mapsto^* R \Rightarrow M' \mapsto^* R$$

and for any $R' \in \{\text{true}, \text{false}, \mathcal{U}\}$,

$$M' \mapsto^* R' \implies M \mapsto^* R'$$

D ELABORATION

LEMMA D.1. *If $A \lesssim B$ then there exist types A_h, D_h, D_l, B_l with*

- (1) $c_l : A \sqsubseteq D_l$ and $c_h : A_h \sqsubseteq D_h$ satisfying $c_l \leq c_h$
- (2) $d_l : B_l \sqsubseteq D_l$ and $d_h : B \sqsubseteq D_h$ satisfying $d_l \leq d_h$
- (3) $e_l : D_l \sqsubseteq D$ and $e_h : D_h \sqsubseteq D$ with $e_l \leq e_h$ where $D = [A] = [B]$.

PROOF. By induction on the proof of $A \lesssim A'$. □

Then the four different choices of cast are all equivalent in the inequational theory:

LEMMA D.2. *Given $A, A_h, B, B_l, D_l, D_h, D, c_l, c_h, d_l, d_h, e_l, e_h$ as in the output of the previous lemma, for any $\Gamma \vdash M : \sigma ! A$, the following four terms are equivalent at type B .*

- (1) $\langle B \leftarrow D_h \rangle \langle D_h \leftarrow A_h \rangle M$
- (2) $\langle B \leftarrow D_h \rangle \langle D_l \leftarrow A \rangle M$
- (3) $\langle B_l \leftarrow D_l \rangle \langle D_l \leftarrow A \rangle M$
- (4) $\langle B \leftarrow D \rangle \langle D \leftarrow A \rangle M$

PROOF. (1) To show (1) is equivalent to (2), it suffices to show

$$\langle D_h \leftarrow A_h \rangle M \equiv \langle D_l \leftarrow A \rangle M$$

which is an instance of the subtyping/cast rule since $c_l \sqsubseteq c_h$.

(2) Similarly to show (2) is equivalent to (3) follows from $d_l \leq d_h$

(3) Lastly we show (4) is equivalent to (2). By cast functoriality,

$$\langle B \leftarrow D \rangle \langle D \leftarrow A \rangle M \equiv \langle B \leftarrow D_h \rangle \langle D_h \leftarrow D \rangle \langle D \leftarrow D_l \rangle \langle D_l \leftarrow A \rangle M$$

And by retraction the middle cast $\langle D_h \leftarrow D \rangle \langle D \leftarrow D_l \rangle$ is the identity. □

D.1 Graduality

$$\text{VAtom } c \quad := \quad \{(V^l, V^r) : \text{val}(V^l) \wedge \text{val}(V^r) \wedge \\ (\Sigma \mid \cdot \mid \cdot \vdash_{\emptyset} V^l : c^l) \wedge (\Sigma \mid \cdot \mid \cdot \vdash_{\emptyset} V^r : c^r)\}$$

$$\text{TAtom } A^l A^r d_{\sigma} \quad := \quad \{(M^l, M^r) : \\ (\Sigma \mid \cdot \mid \cdot \vdash_{d_{\sigma}^l} M^l : A^l) \wedge (\Sigma \mid \cdot \mid \cdot \vdash_{d_{\sigma}^r} M^r : A^r)\}$$

$$\text{ECtxAtom } c (\sigma^l ! A^l) (\sigma^r ! A^r) \quad := \quad \{(x^l.M^l, x^r.M^r) : \\ (\Sigma \mid x^l : c^l \mid \cdot \vdash_{\sigma^l} M^l : A^l) \wedge (\Sigma \mid x^r : c^r \mid \cdot \vdash_{\sigma^r} M^r : A^r)\}$$

Fig. 21. Well typed atoms

Our main goal is to prove the soundness of the inequational theory with respect to the logical relation. That is

THEOREM D.3 (GRADUALITY). *If $\Gamma^{\square} \vdash_{d_{\sigma}} M \sqsubseteq N : c$ then $\Gamma^{\square} \vDash_{d_{\sigma}} M \sqsubseteq N : c$*

PROOF. By induction on the term precision derivation.

- (1) (ValSubst) Lemma D.31
- (2) (MonadUnitL) Lemma D.32
- (3) (MonadUnitR) Lemma D.33
- (4) (MonadAssoc) Lemma D.34
- (5) (BoolBeta) Lemmas D.36 and D.37
- (6) (BoolEta) Lemma D.35
- (7) (IfEval) Lemma D.38
- (8) (FunBeta) Lemma D.39
- (9) (FunEta) Lemma D.40
- (10) (AppEval) Lemma D.41
- (11) (HandleBetaRet) Lemma D.42
- (12) (HandleBetaRaise) Lemma D.43
- (13) (HandleEmpty) Lemma D.45
- (14) (HandleExt) Lemma D.46
- (15) (RaiseEval) Lemma D.44
- (16) (Variable) Lemma D.23
- (17) (Let) Lemma D.27
- (18) (Boolean) Lemma D.22
- (19) (If) Lemma D.26
- (20) (Lambda) Lemma D.24
- (21) (App) Lemma D.25
- (22) (Raise) Lemma D.28
- (23) (HandleCong) Lemma D.29
- (24) (Transitivity) Lemma D.69
- (25) (ErrBot) Lemma D.47
- (26) (ErrStrict) Lemma D.48
- (27) (SubtyMon) Lemma D.49
- (28) (ValUpSub) Lemma D.61
- (29) (ValDnSub) Lemma D.61

- (30) (EffUpSub) Lemma D.61
- (31) (EffDnSub) Lemma D.61
- (32) (ValUpL) Follows from Lemma D.51.
- (33) (ValUpR) Follows from Lemma D.50.
- (34) (ValUpEval) Lemma D.58
- (35) (ValDnR) Follows from Lemma D.53.
- (36) (ValDnL) Follows from Lemma D.52.
- (37) (ValDnEval) Lemma D.59
- (38) (ValRetract) Lemma D.60.
- (39) (EffUpL) Follows from Lemma D.55
- (40) (EffUpR) Follows from Lemma D.54
- (41) (EffDnR) Follows from Lemma D.57
- (42) (EffDnL) Follows from Lemma D.56
- (43) (EffRetract) Lemma D.60.

□

We begin with a few lemmas that will be useful in our proofs.

D.1.1 Lemmas.

LEMMA D.4. *If $(V_1, V_2) \in R$, and V_1 and V_2 are values of type A^l and A^r respectively, then $(V_1, V_2) \in \mathcal{R}_j^{\sim} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$.*

PROOF. We will establish the first disjunct in the definition of $\mathcal{R}^{\sim} \llbracket \cdot \rrbracket$. This follows by assumption. □

LEMMA D.5. *If $(V_1, V_2) \in \mathcal{R}_j^{\sim} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$, then $(V_1, V_2) \in \mathcal{E}_j^{\sim} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$.*

PROOF. Let $\sim \in \{<, >\}$, and suppose $(V_1, V_2) \in \mathcal{R}_j^{\sim} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$. Notice that regardless of whether \sim is $<$ or $>$, we will be able to show the last clause in the definition of $\mathcal{E}_j^{\sim} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$ or $\mathcal{E}_j^{\geq} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$. In particular, we can take $k = j$, $V_1 = V_1$, and $V_2 = V_2$, noting that V_1 steps to itself in 0 steps, as does V_2 . Thus, it remains to show that V_1 and V_2 are related by $\mathcal{R}_j^{\leq} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$ or $\mathcal{R}_j^{\geq} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$. This is true by assumption. □

LEMMA D.6. *If $(V_1, V_2) \in \mathcal{V}_j^{\sim} \llbracket c \rrbracket$, then $(V_1, V_2) \in \mathcal{E}_j^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket$.*

PROOF. By Lemma D.5 (with $R = \mathcal{V}^{\sim} \llbracket c \rrbracket$), it suffices to show that $(\sigma_1 V_1, \sigma_1 V_2) \in \mathcal{R}_j^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket$. This is true by Lemma D.4, again with $R = \mathcal{V}^{\sim} \llbracket c \rrbracket$. □

LEMMA D.7 (ANTI-REDUCTION, ONE-SIDED). *Suppose $M_1 \mapsto^{i_1} M'_1$ and $M_2 \mapsto^{i_2} M'_2$.*

If $(M'_1, M'_2) \in \mathcal{E}_{j-i_2}^{\geq} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$, then $(M_1, M_2) \in \mathcal{E}_j^{\geq} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$.

Similarly, if $(M'_1, M'_2) \in \mathcal{E}_{j-i_1}^{\leq} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$, then $(M_1, M_2) \in \mathcal{E}_j^{\leq} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$.

PROOF. We prove the first statement; the second is analogous (and in fact easier). The assumption that $(M'_1, M'_2) \in \mathcal{E}_{j-i_2}^{\geq} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$ has four cases:

- (1) $M'_2 \mapsto^{j-i_2+1}$. In this case, $M_2 \mapsto^{i_2} M'_2 \mapsto^{j-i_2+1}$, i.e. $M_2 \mapsto^{j+1}$. Thus, we may assert the first disjunct in the definition of $\mathcal{E}_j^{\geq} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$.
- (2) There exists $k \leq j - i_2$ such that $M'_1 \mapsto^{j-i_2-k} \mathcal{U}$, and furthermore $M'_1 \mapsto^* \mathcal{U}$. In this case, we have that $M_2 \mapsto^{i_2} M'_2 \mapsto^{j-i_2-k} \mathcal{U}$, so $M_2 \mapsto^{j-k} \mathcal{U}$. Also, $M_1 \mapsto^{i_1} M'_1 \mapsto^* \mathcal{U}$, so $M_1 \mapsto^* \mathcal{U}$. Thus, we may assert the second disjunct.

- (3) There exists $k \leq j - i_2$ and N_2 such that $M'_2 \mapsto^{j-i_2-k} N_2$ and $M'_1 \mapsto^* \mathcal{U}$. In this case we have $M_2 \mapsto^{i_2} M'_2 \mapsto^{j-i_2-k} N_2$, so $M_2 \mapsto^{j-k} N_2$. Thus, we may assert the third disjunct.
- (4) Similar to previous case. □

LEMMA D.8 (ANTI-REDUCTION). *Suppose $M_1 \mapsto^{i_1} M'_1$ and $M_2 \mapsto^{i_2} M'_2$, and that $(M'_1, M'_2) \in \mathcal{E}_{j-m}^{\sim} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$, where $m = \min\{i_1, i_2\}$. Then $(M_1, M_2) \in \mathcal{E}_j^{\sim} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$.*

PROOF. Follows from one-sided anti-reduction (Lemma D.7) and downward closure. □

LEMMA D.9 (FORWARD REDUCTION, ONE-SIDED). *Suppose $M_1 \mapsto^{i_1} M'_1$ and $M_2 \mapsto^{i_2} M'_2$. If $(M_1, M_2) \in \mathcal{E}_{j+i_2}^{\geq} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$, then $(M'_1, M'_2) \in \mathcal{E}_j^{\geq} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$.*

Similarly, if $(M_1, M_2) \in \mathcal{E}_{j+i_1}^{\leq} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$, then $(M'_1, M'_2) \in \mathcal{E}_j^{\leq} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$.

PROOF. Follows from determinism of evaluation and a case analysis on the assumption that M_1 and M_2 are related. □

LEMMA D.10 (FORWARD REDUCTION). *Suppose $M_1 \mapsto^{i_1} M'_1$ and $M_2 \mapsto^{i_2} M'_2$, and that $(M_1, M_2) \in \mathcal{E}_{j+m} \llbracket d_\sigma \rrbracket (R, A^l, A^r)$, where $m = \max\{i_1, i_2\}$. Then $(M'_1, M'_2) \in \mathcal{E}_j \llbracket d_\sigma \rrbracket (R, A^l, A^r)$.*

PROOF. Follows from one-sided forward reduction (Lemma D.9) and downward closure. □

Frequently in our proofs we will encounter a situation where we know that two evaluation contexts are related in the $\mathcal{K}^{\sim} \llbracket \cdot \rrbracket$ relation, that is, substituting related values gives related outputs. On the other hand, as a cast applied to a value is not necessarily itself a value, we cannot reason directly about what happens when such semantic values are substituted into related evaluation contexts. We therefore introduce the following lemma.

LEMMA D.11. *Suppose E_1 and E_2 are evaluation contexts that take values to values. Let V_1 and V_2 be values (not necessarily related) such that*

$$(E_1[V_1], E_2[V_2]) \in \mathcal{E}_j^{\sim} \llbracket d'_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket.$$

Furthermore, let $(E^l[x^l], E^r[x^r]) \in \mathcal{K}_j^{\sim} \llbracket c \rrbracket \mathcal{E}^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket d \rrbracket$.

Then

$$(E^l[E_1[V_1]], E^r[E_2[V_2]]) \in \mathcal{E}_j^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket d \rrbracket.$$

PROOF. We show the proof for $\sim = >$.

By assumption, we have that there exist values V'_1 and V'_2 such that $E_1[V_1] \mapsto^{i_1} V'_1$ and $E_2[V_2] \mapsto^{i_2} V'_2$, for some i_1 and i_2 .

Thus, $E^l[E_1[V_1]] \mapsto^{i_1} E^l[V'_1]$ and likewise $E^r[E_2[V_2]] \mapsto^{i_2} E^r[V'_2]$.

By one-sided anti-reduction (Lemma D.7), it suffices to show that

$$(E^l[V'_1], E^r[V'_2]) \in \mathcal{E}_{j-i_2}^{\geq} \llbracket d_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket d \rrbracket.$$

By assumption on E^l and E^r being related, it suffices to show that $(V'_1, V'_2) \in \mathcal{V}_{j-i_2}^{\geq} \llbracket c \rrbracket$.

Now by one-sided forward reduction (Lemma D.9), it suffices to show

$$(E_1[V_1], E_2[V_2]) \in \mathcal{E}_{(j-i_2)+i_2}^{\geq} \llbracket d_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c \rrbracket.$$

But this is precisely our assumption, so we are finished. □

Remark: The reason why we needed to consider cases on \sim separately is that the more “generic”/two-sided anti-reduction and forward-reduction lemmas involve the min or max of the number of steps taken by the two terms. These may not be equal, in which case the arithmetic wouldn't work out. But this doesn't mean the above lemma is false. Conceptually, what is happening is that in the two-sided variants of the lemmas, \sim could be either $>$ or $<$. On the other hand, the key here is that \sim stays the same throughout the application of anti-reduction and forward reduction, so we are able to use the more specific, one-sided lemmas.

LEMMA D.12 (TIME-OUT). *If $M_1 \mapsto^{(i+1)}$, then $(M_1, M_2) \in \mathcal{E}_i^{\leq} \llbracket d_\sigma \rrbracket R$. Similarly, if $M_2 \mapsto^{(i+1)}$, then $(M_1, M_2) \in \mathcal{E}_i^{\geq} \llbracket d_\sigma \rrbracket R$.*

PROOF. Suppose $M_1 \mapsto^{(i+1)}$. Then we may assert the first disjunct in the definition of $\mathcal{E}_i^{\leq} \llbracket d_\sigma \rrbracket R$ to conclude that $(M_1, M_2) \in \mathcal{E}_i^{\leq} \llbracket d_\sigma \rrbracket R$. Likewise, if $M_2 \mapsto^{(i+1)}$, then we may assert the first disjunct in the definition of $\mathcal{E}_i^{\geq} \llbracket d_\sigma \rrbracket R$ to conclude that $(M_1, M_2) \in \mathcal{E}_i^{\geq} \llbracket d_\sigma \rrbracket R$. \square

We present two trivial lemmas about the later modality. We do this to cut down on tedious reasoning about step indices within other proofs.

LEMMA D.13. *Let R be a monotone step-indexed relation. If $(M_1, M_2) \in R_j$, then $(M_1, M_2) \in (\blacktriangleright R)_j$.*

PROOF. Suppose $(M_1, M_2) \in R_j$. If $j = 0$, then $(M_1, M_2) \in (\blacktriangleright R)_0$ trivially.

Otherwise, let $j = j' + 1$. By monotonicity of R , we have $(M_1, M_2) \in R_{j'}$, from which it follows that $(M_1, M_2) \in (\blacktriangleright R)_j$. \square

LEMMA D.14. *Let R be a monotone step-indexed relation, and let j be of the form $j = j' + 1$. If $(M_1, M_2) \in (\blacktriangleright R)_j$, then $(M_1, M_2) \in (\blacktriangleright R)_{j'}$.*

PROOF. Suppose $(M_1, M_2) \in (\blacktriangleright R)_j$. Since $j = j' + 1$, by definition of \blacktriangleright we must have that $(M_1, M_2) \in R_{j'}$. By the previous lemma (Lemma D.13), we conclude $(M_1, M_2) \in (\blacktriangleright R)_{j'}$, which is what we needed to show. \square

LEMMA D.15 (REASONING WITH “LATER” WHEN BOTH SIDES STEP). *Suppose $M \mapsto^1 M'$ and $N \mapsto^1 N'$, and that $(M', N') \in (\blacktriangleright \mathcal{E}_k^- \llbracket d_\sigma \rrbracket)_k R$. Then $(M, N) \in \mathcal{E}_k^- \llbracket d_\sigma \rrbracket R$.*

PROOF. First suppose $k = 0$. Then by the time-out lemma (Lemma D.12), regardless of whether \sim is $<$ or $>$, we have $(M, N) \in \mathcal{E}_0^- \llbracket d_\sigma \rrbracket R$.

Now suppose $k \geq 1$. Then by the definition of later, we have that $(M', N') \in \mathcal{E}_{k-1}^- \llbracket d_\sigma \rrbracket R$, so by anti-reduction we have that $(M, N) \in \mathcal{E}_k^- \llbracket d_\sigma \rrbracket R$. \square

LEMMA D.16 (LÖB-INDUCTION). *Let $P(n)$ be a predicate indexed by a natural number n . Suppose for all natural numbers n , we have that $(\blacktriangleright^m P)(n)$ implies $P(n)$ for all $m \geq 1$. Then $P(n)$ is true for all natural numbers n .*

PROOF. The proof is by induction on n . When $n = 0$, the assumption says that $(\blacktriangleright P)(0)$ implies $P(0)$ (we have taken $m = 1$). So, it suffices to show that $(\blacktriangleright P)(0)$ holds. This is true by the definition of later.

Now let $n \geq 1$ be fixed, and suppose $P(n)$ is true. We claim that $P(n+1)$ is true. By our assumption, it will suffice to show that $(\blacktriangleright P)(n+1)$ is true. (We have again chosen $m = 1$.) By definition of later, we must show $P(n)$ is true. But $P(n)$ is true by assumption. \square

We now introduce a key lemma about evaluation contexts.

Note: In the below, we omit explicit mention of the types associated to the relations that parameterize $\mathcal{E}^- \llbracket \cdot \rrbracket$ and $\mathcal{R}^- \llbracket \cdot \rrbracket$.

LEMMA D.17. *If*

- (1) $(M_1, M_2) \in \mathcal{E}_j^\sim \llbracket d'_\sigma \rrbracket S'$
- (2) *For all* $k \leq j$ *and* $(N_1, N_2) \in \mathcal{R}_k^\sim \llbracket d'_\sigma \rrbracket S'$, *we have* $(E_1[N_1], E_2[N_2]) \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket S$,

then $(E_1[M_1], E_2[M_2]) \in \mathcal{E}_j^\sim \llbracket d_\sigma \rrbracket S$.

PROOF. We prove the lemma for $\sim = >$; the other case is similar. Based on assumption (1), there are four cases:

- (1) Case $M_2 \mapsto^{j+1}$. We have $E_2[M_2] \mapsto^{j+1}$, so we may assert the first disjunct in the definition of $\mathcal{E}_j^\sim \llbracket d_\sigma \rrbracket S$ to conclude that $(E_1[M_1], E_2[M_2]) \in \mathcal{E}_j^\sim \llbracket d_\sigma \rrbracket S$.
- (2) Case $\exists k \leq j$ such that $M_2 \mapsto^{j-k} \mathcal{U}$ and $M_1 \mapsto^* \mathcal{U}$. We have $E_2[M_2] \mapsto^{j-k+1} \mathcal{U}$. If $k = 0$, then we have $E_2[M_2] \mapsto^{j+1}$, so we may assert the first disjunct. Otherwise, if $k \geq 1$, then we may take $k' = k - 1$ and observe that $E_2[M_2] \mapsto^{j-k'} \mathcal{U}$.
- (3) Case $\exists k \leq j, \exists V_2$ such that $M_2 \mapsto^{j-k} N_2$ and $M_1 \mapsto^* \mathcal{U}$. We have $E_2[M_2] \mapsto^{j-k} E_2[N_2]$, so we may assert the third disjunct with $k = k$ and $N_2 = E_2[N_2]$.
- (4) Case $\exists k \leq j, \exists (N_1, N_2) \in \mathcal{R}_k^\sim \llbracket d_\sigma \rrbracket S'$ such that $M_2 \mapsto^{j-k} N_2$ and $M_1 \mapsto^* N_1$. We have $E_1[M_1] \mapsto^{i_1} E_1[N_1]$ for some i_1 , and $E_2[M_2] \mapsto^{j-k} E_2[N_2]$. By assumption (2), we have $(E_1[N_1], E_2[N_2]) \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket S$. Thus, we may assert the fourth disjunct with $V_1 = E_1[N_1]$ and $V_2 = E_2[N_2]$.

□

LEMMA D.18 (“SEMANTIC BIND”). *Let* $c : A \sqsubseteq A'$ *and* $d : B \sqsubseteq B'$. *Let* E_1 *and* E_2 *be evaluation contexts such that* $\Sigma \mid \Gamma \mid \bullet : (d_\sigma^l ! A) \vdash_{d_\sigma^l} E_1 : B$ *and* $\Sigma \mid \Gamma \mid \bullet : (d_\sigma^r ! A') \vdash_{d_\sigma^r} E_2 : B'$. *Suppose*

- (1) $(M_1, M_2) \in \mathcal{E}_j^\sim \llbracket d'_\sigma \rrbracket (S', A, A')$.
- (2) *For all* $k \leq j$ *and* $(V_1, V_2) \in S'_k$, *we have* $(E_1[V_1], E_2[V_2]) \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket (S, B, B')$.
- (3) *For all* $k \leq j$ *and for all* $\epsilon : c_\epsilon \rightsquigarrow d_\epsilon \in d'_\sigma$, *if* E_1 *catches* ϵ *or* E_2 *catches* ϵ , *then for all* $V^l, V^r \in (\blacktriangleright \mathcal{V}^\sim \llbracket c_\epsilon \rrbracket)_k$ *and all evaluation contexts* $E^l \# \epsilon$ *and* $E^r \# \epsilon$ *such that* $(x^l.E^l[x^l], x^r.E^r[x^r]) \in (\blacktriangleright \mathcal{K}^\sim \llbracket d_\epsilon \rrbracket)_k (\mathcal{E}^\sim \llbracket d'_\sigma \rrbracket (S', A, A'), (d_\sigma^l ! A), (d_\sigma^r ! A'))$, *we have*
 $(E_1[E^l[\text{raise } \epsilon @ c_\epsilon^l \rightsquigarrow d_\epsilon^l(V^l)], E_2[E^r[\text{raise } \epsilon @ c_\epsilon^r \rightsquigarrow d_\epsilon^r(V^r)]]) \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket (S, B, B')$.

Then $(E_1[M_1], E_2[M_2]) \in \mathcal{E}_j^\sim \llbracket d_\sigma \rrbracket (S, B, B')$.

PROOF. We use Löb induction (Lemma D.16). We assume that if the premises of the lemma are satisfied “later”, then the conclusion holds later. We show under this assumption that the lemma holds “now”.

We first apply Lemma D.17. The first hypothesis is immediate. Now let $k \leq j$ and let $(N_1, N_2) \in \mathcal{R}_k^\sim \llbracket d'_\sigma \rrbracket (S', A, A')$. We need to show that

$$(E_1[N_1], E_2[N_2]) \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket (S, B, B').$$

There are two cases to consider. In the first case, N_1 and N_2 are values and $(N_1, N_2) \in \mathcal{V}_j^\sim \llbracket c \rrbracket$. Then by assumption (2) with $k = j$, we have $(E_1[N_1], E_2[N_2]) \in \mathcal{E}_j^\sim \llbracket d_\sigma \rrbracket (S, B, B')$, as needed.

In the second case, there exist $\epsilon' : c' \rightsquigarrow d' \in d'_\sigma$, $E^l \# \epsilon'$, $E^r \# \epsilon'$, and V^l, V^r such that $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^\sim \llbracket c' \rrbracket)_j$, and $(x^l.E^l[x^l], x^r.E^r[x^r]) \in (\blacktriangleright \mathcal{K}^\sim \llbracket d' \rrbracket)_j (\mathcal{E}^\sim \llbracket d'_\sigma \rrbracket (S', A, A'), (d_\sigma^l ! A), (d_\sigma^r ! A'))$, and $N_1 = E^l[\text{raise } \epsilon'(V^l)]$ and $N_2 = E^r[\text{raise } \epsilon'(V^r)]$. Let $N'_1 = E_1[N_1] = E_1[E^l[\text{raise } \epsilon'(V^l)]]$ and $N'_2 = E_2[N_2] = E_2[E^r[\text{raise } \epsilon'(V^r)]]$.

We need to show that

$$(N'_1, N'_2) \in \mathcal{E}_j^\sim \llbracket d_\sigma \rrbracket (S, B, B').$$

We now consider whether one of E_1 or E_2 catches ϵ' , or whether neither catches it. In the former case, assumption (3) immediately implies the desired result.

Now suppose neither E_1 nor E_2 catches ϵ . In this case, note that since $\epsilon' \# E^l$ and $\epsilon' \# E_1$, we have $\epsilon' \# E_1[E^l]$. Likewise, we have $\epsilon' \# E_2[E^r]$. It follows that N'_1 and N'_2 are stuck terms, i.e., they do not step. Thus, it suffices to show that

$$(N'_1, N'_2) \in \mathcal{R}_j \llbracket d_\sigma \rrbracket (S, B, B').$$

We first claim $(V^l, V^r) \in (\blacktriangleright \mathcal{V} \llbracket c' \rrbracket)_j$. Since $(V^l, V^r) \in (\blacktriangleright \mathcal{V} \llbracket c \rrbracket)_j$, this follows by Lemma D.14.

We now claim that

$$(x^l.(E_1[E^l[x^l]]), x^r.(E_2[E^r[x^r]])) \in (\blacktriangleright \mathcal{K} \llbracket d' \rrbracket)_j (\mathcal{E} \llbracket d_\sigma \rrbracket (S, B, B'), (d'_\sigma ! B), (d'_\sigma ! B')).$$

To this end, let $k \leq j$ and let $(V^l, V^r) \in (\blacktriangleright \mathcal{V} \llbracket d' \rrbracket)_k$. We need to show that

$$(E_1[E^l[V^l]], E_2[E^r[V^r]]) \in (\blacktriangleright \mathcal{E} \llbracket d_\sigma \rrbracket)_k (S, B, B').$$

By the Löb induction hypothesis, it suffices to show that the three hypotheses of the lemma hold later. We claim that $(E^l[V^l], E^r[V^r]) \in (\mathcal{E}_k \llbracket d'_\sigma \rrbracket \mathcal{V} \llbracket c \rrbracket)$. To see this, recall our assumption that

$$(x^l.E^l[x^l], x^r.E^r[x^r]) \in (\blacktriangleright \mathcal{K} \llbracket d' \rrbracket)_j (\mathcal{E} \llbracket d'_\sigma \rrbracket \mathcal{V} \llbracket c \rrbracket).$$

Thus, we have that $(E^l[V^l], E^r[V^r]) \in (\blacktriangleright \mathcal{E} \llbracket d'_\sigma \rrbracket)_k (\mathcal{V} \llbracket c \rrbracket)$, which is what we needed to show. \square

We now introduce a few lemmas about precision derivations. We first show how we may “compose” precision derivations:

LEMMA D.19 (CUT ADMISSIBILITY FOR PRECISION DERIVATIONS). \bullet If $c : A \sqsubseteq B$ and $d : B \sqsubseteq C$ then $c \circ d : A \sqsubseteq C$.
 \bullet If $d_\sigma : \sigma \sqsubseteq \sigma'$ and $d'_\sigma : \sigma' \sqsubseteq \sigma''$ then $d_\sigma \circ d'_\sigma : \sigma \sqsubseteq \sigma''$.

PROOF. We prove these statements simultaneously by induction on d and d'_σ .

- Case $d = \text{bool}$. We have $B = C = \text{bool}$, so $c = \text{bool}$ (the reflexivity derivation). Thus, we may take $c \circ d = \text{bool}$.
- Case $d = d_i \rightarrow_{d_\sigma} d_o$. Inspecting the rules in figure 14, we see that $B = B_i \rightarrow_{B_\sigma} B_o$ and $C = C_i \rightarrow_{C_\sigma} C_o$. Thus, we must have $A = A_i \rightarrow_{A_\sigma} A_o$, which means that $c = c_i \rightarrow_{c_\sigma} c_o$. We may take $c \circ d = (c_i \circ d_i) \rightarrow_{c_\sigma \circ d_\sigma} (c_o \circ d_o)$. By our inductive hypotheses, we have (1) $c_i \circ d_i : A_i \sqsubseteq C_i$, (2) $c_\sigma \circ d_\sigma : A_\sigma \sqsubseteq C_\sigma$, and (3) $c_o \circ d_o : A_o \sqsubseteq C_o$. Now, using the type precision formation rule for functions, we get that $(c_i \circ d_i) \rightarrow_{c_\sigma \circ d_\sigma} (c_o \circ d_o) : (A_i \rightarrow_{A_\sigma} A_o) \sqsubseteq (C_i \rightarrow_{C_\sigma} C_o)$.
- Case $d'_\sigma = ?$. Define $? \circ ? = ?$. Define $\text{Inj}(d) \circ ? = \text{Inj}(d)$. An concrete effect set cannot be composed with $?$.
- Case $d'_\sigma = \text{Inj}(d)$. Note that $\sigma'' = ?$. We define $d_\sigma \circ \text{Inj}(d) = \text{Inj}(d_\sigma \circ d)$.
- Case $d'_\sigma = d'_c$: Define $(d_c \circ d'_c)$ by $\epsilon : c \rightsquigarrow d \in (d_c \circ d'_c)$ if and only if $c = c_1 \circ c_2$ and $d = d_1 \circ d_2$ with $\epsilon : c \rightsquigarrow_1 d_1 \in d_c$ and $\epsilon : c \rightsquigarrow_2 d_2 \in d'_c$.

\square

LEMMA D.20 (REFLEXIVITY OF COMPOSITION). Let $c : A \sqsubseteq B$ and $d_\sigma : \sigma \sqsubseteq \sigma'$. The following hold.

- $c \circ B = A \circ c = c$.

- $d_\sigma \circ \sigma' = \sigma \circ d_\sigma = d_\sigma$.

PROOF. Follows from the uniqueness of precision derivations. That is, $c \circ B$, $A \circ c$, and c all are all proofs of $A \sqsubseteq B$, hence are equal. \square

LEMMA D.21 (DECOMPOSITION). *Suppose $\epsilon@c \rightsquigarrow d \in d_\sigma \circ d'_\sigma$. Then there exist c_1, c_2 and d_1, d_2 such that $\epsilon@c_1 \rightsquigarrow d_1 \in d_\sigma$ and $\epsilon@c_2 \rightsquigarrow d_2 \in d'_\sigma$ and $c = c_1 \circ c_2$ and $d = d_1 \circ d_2$.*

PROOF. By induction on d'_σ .

- Case $d'_\sigma = ?$. If $d_\sigma = ?$, then our assumption becomes $\epsilon@c \rightsquigarrow d \in ? \circ ? = ?$. By definition of membership in $?$, this means that $\epsilon@c^r \rightsquigarrow d^r \in \Sigma$.

We may take $c_1 = c$ and take c_2 to be the reflexivity derivation for $c^r \sqsubseteq c^r$. Likewise, we take $d_1 = d$ and d_2 to be the reflexivity derivation for $d^r \sqsubseteq d^r$. Note that $\epsilon@c_2 \rightsquigarrow d_2 \in ?$, because $c_2^r = c^r$ and $d_2^r = d^r$, and we know $\epsilon@c^r \rightsquigarrow d^r \in \Sigma$. We also have that $c = c_1 \circ c_2$ and $d = d_1 \circ d_2$, using Lemma D.20.

If $d_\sigma = \text{inj}(d_\sigma)$, then our assumption becomes $\epsilon@c \rightsquigarrow d \in \text{inj}(d_\sigma)$. By definition of membership in $\text{Inj}(\cdot)$, we have that $\epsilon@c \rightsquigarrow d \in d_\sigma$. We may again take $c_1 = c$ and c_2 to be the reflexivity derivation for $c^r \sqsubseteq c^r$, and likewise for d_1 and d_2 . The same reasoning as above applies.

- Case $d'_\sigma = \text{inj}(d_\sigma)$. By definition of composition, our assumption becomes $\epsilon@c \rightsquigarrow d \in (d_\sigma \circ \text{inj}(d_\sigma)) = \text{inj}(d_\sigma \circ d_\sigma)$.

By the induction hypothesis, there are c_1, c_2 and d_1, d_2 such that $\epsilon@c_1 \rightsquigarrow d_1 \in d_\sigma$ and $\epsilon@c_2 \rightsquigarrow d_2 \in d_\sigma$ and $c = c_1 \circ c_2$ and $d = d_1 \circ d_2$. By definition of membership in $\text{Inj}(\cdot)$, we have $\epsilon@c_2 \rightsquigarrow d_2 \in \text{inj}(d_\sigma) = d'_\sigma$.

- Case $d'_\sigma = d'_c$ (concrete effect set). Similar to previous case.

\square

D.1.2 *Congruence Rules.* With these lemmas, we can prove the soundness of the term precision congruence rules. The proofs are by induction on the term precision derivation.

LEMMA D.22 (CONGRUENCE FOR BOOLEANS).

PROOF. We need to show that $\Gamma^\sqsubseteq \vDash_{d_\sigma} \llbracket \text{true} \rrbracket \sqsubseteq \llbracket \text{true} \rrbracket \in \text{bool}$, and likewise for `false` (we will show this for `true` only; the reasoning for `false` is exactly the same.)

Let $\sim \in \{<, >\}$ and let $(\gamma_1, \gamma_2) \in \mathcal{G}_i^\sim \llbracket \Gamma^\sqsubseteq \rrbracket$. We need to show

$$(\text{true}[\gamma_1], \text{true}[\gamma_2]) \in \mathcal{E}_i^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket \text{bool} \rrbracket,$$

i.e.,

$$(\text{true}, \text{true}) \in \mathcal{E}_i^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket \text{bool} \rrbracket.$$

By Lemma D.6, it suffices to show that $(\text{true}, \text{true}) \in \mathcal{V}_i^\sim \llbracket \text{bool} \rrbracket$. This is true according to the definition of the logical relation. \square

LEMMA D.23 (CONGRUENCE FOR VARIABLES).

PROOF. We need to show that $\Gamma^\sqsubseteq, x_1 \sqsubseteq x_2 : c, \Gamma'^\sqsubseteq \vDash_{d_\sigma} x_1 \sqsubseteq x_2 \in c$.

Let $\sim \in \{<, >\}$, and let $\widehat{\Gamma}^\sqsubseteq = \Gamma^\sqsubseteq, x_1 \sqsubseteq x_2 : c, \Gamma'^\sqsubseteq$. Let $(\gamma_1, \gamma_2) \in \mathcal{G}_i^\sim \llbracket \widehat{\Gamma}^\sqsubseteq \rrbracket$. We need to show

$$(x_1[\gamma_1], x_2[\gamma_2]) \in \mathcal{E}_i^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket.$$

By Lemma D.6, it suffices to show that $(\gamma_1(x_1), \gamma_2(x_2)) \in \mathcal{V}_i^\sim \llbracket c \rrbracket$. But this follows from the fact that $(\gamma_1, \gamma_2) \in \mathcal{G}_i^\sim \llbracket \widehat{\Gamma}^\square \rrbracket$. In particular, by the definition of the logical relation, since $(x_1 \sqsubseteq x_2 : c) \in \widehat{\Gamma}^\square$, we have $(\gamma_1(x_1), \gamma_2(x_2)) \in \mathcal{V}_i^\sim \llbracket c \rrbracket$. \square

LEMMA D.24 (CONGRUENCE FOR LAMBDA).

PROOF. Suppose $\Gamma^\square, x \sqsubseteq y : c \vDash_{d_\sigma} M \sqsubseteq N \in d$. We need to show that $\Gamma^\square \vDash_{d_\sigma} \lambda x.M \sqsubseteq \lambda y.N \in c \rightarrow_{d_\sigma} d$.

Let $\sim \in \{<, >\}$ and let $(\gamma_1, \gamma_2) \in \mathcal{G}_i^\sim \llbracket \Gamma^\square \rrbracket$. We need to show

$$((\lambda x.M)[\gamma_1], (\lambda y.N)[\gamma_2]) \in \mathcal{E}_i^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rightarrow_{d_\sigma} d \rrbracket.$$

Let $V_1 = \lambda x.M[\gamma_1]$ and $V_2 = \lambda y.N[\gamma_2]$. By Lemma D.6, it will suffice to show that $(V_1, V_2) \in \mathcal{V}_i^\sim \llbracket c \rightarrow_{d_\sigma} d \rrbracket$. To this end, let $k \leq i$ and let $(V_{i1}, V_{i2}) \in \mathcal{V}_k^\sim \llbracket c \rrbracket$. We will show that $(V_1 V_{i1}, V_2 V_{i2}) \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket d \rrbracket$.

Let $M' = (M[\gamma_1])(V_{i1}/x)$ and let $N' = (N[\gamma_2])(V_{i2}/y)$. Note that $(V_1 V_{i1}) \mapsto^1 M'$, and similarly $(V_2 V_{i2}) \mapsto^1 N'$. Thus, if $k = 0$, then by the Time-out Lemma (Lemma D.12), we conclude that $(V_1 V_{i1}, V_2 V_{i2}) \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket d \rrbracket$.

Hence, from now on, we assume $k \geq 1$. By the Anti-reduction lemma (Lemma D.8) (with $i_1 = i_2 = 1$ and $j = k$), it will suffice to show that $(M', N') \in \mathcal{E}_{k-1}^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket d \rrbracket$.

This will follow by our inductive hypothesis, which says that for any $\sim \in \{<, >\}$, any natural number n , and any $(\gamma'_1, \gamma'_2) \in \mathcal{G}_n^\sim \llbracket \Gamma^\square, x \sqsubseteq y : c \rrbracket$, we have

$$(M[\gamma'_1], N[\gamma'_2]) \in \mathcal{E}_n^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket d \rrbracket.$$

Let $\gamma'_1 = \gamma_1, V_{i1}/x$, let $\gamma'_2 = \gamma_2, V_{i2}/y$. It is easily verified that $(\gamma'_1, \gamma'_2) \in \mathcal{G}_{k-1}^\sim \llbracket \Gamma^\square, x \sqsubseteq y : c \rrbracket$. (Doing so requires the monotonicity lemma, combined with the fact that $(\gamma_1, \gamma_2) \in \mathcal{G}_i^\sim \llbracket \Gamma^\square \rrbracket$ and that $k-1 < k \leq i$). Taking $n = k-1$ above, and noting that $M' = M[\gamma'_1]$ and $N' = N[\gamma'_2]$, it follows that $(M', N') \in \mathcal{E}_{k-1}^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket d \rrbracket$, as we wanted to show. \square

LEMMA D.25 (CONGRUENCE FOR FUNCTION APPLICATION).

PROOF. Suppose $\Gamma^\square \vDash_{d_\sigma} M_1 \sqsubseteq M_2 \in c \rightarrow_{d_\sigma} d$, and that $\Gamma^\square \vDash_{d_\sigma} N_1 \sqsubseteq N_2 \in c$.

We need to show that $\Gamma^\square \vDash_{d_\sigma} M_1 N_1 \sqsubseteq M_2 N_2 \in d$.

Let $\sim \in \{<, >\}$ and let $(\gamma_1, \gamma_2) \in \mathcal{G}_i^\sim \llbracket \Gamma^\square \rrbracket$. We need to show

$$(M_1 N_1[\gamma_1], M_2 N_2[\gamma_2]) \in \mathcal{E}_i^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket d \rrbracket.$$

By Lemma D.18, it will suffice to show that

(1) $(M_1[\gamma_1], M_2[\gamma_2]) \in \mathcal{E}_i^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rightarrow_{d_\sigma} d \rrbracket$, and that (2) for all $k \leq i$ and $(V_1, V_2) \in \mathcal{V}_k^\sim \llbracket c \rightarrow_{d_\sigma} d \rrbracket$, we have $(V_1 N_1, V_2 N_2) \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket d \rrbracket$.

(1) follows immediately from our first top-level assumption.

To show (2), we again apply Lemma D.18. It follows from our second top-level assumption that $(N_1[\gamma_1], N_2[\gamma_2]) \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket$. Now let $k' \leq k$ and $(V'_1, V'_2) \in \mathcal{V}_{k'}^\sim \llbracket c \rrbracket$. We claim that

$$(V_1 V'_1, V_2 V'_2) \in \mathcal{E}_{k'}^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket d \rrbracket.$$

This holds since $(V_1, V_2) \in \mathcal{V}_k^\sim \llbracket c \rightarrow_{d_\sigma} d \rrbracket$ and $(V'_1, V'_2) \in \mathcal{V}_{k'}^\sim \llbracket c \rrbracket$. \square

LEMMA D.26 (CONGRUENCE FOR IF).

PROOF. Suppose:

- (1) $\Gamma^{\sqsubseteq} \vDash_{d_\sigma} \llbracket M \rrbracket \sqsubseteq \llbracket M' \rrbracket \in \text{bool}$
- (2) $\Gamma^{\sqsubseteq} \vDash_{d_\sigma} \llbracket N_t \rrbracket \sqsubseteq \llbracket N'_t \rrbracket \in c$
- (3) $\Gamma^{\sqsubseteq} \vDash_{d_\sigma} \llbracket N_f \rrbracket \sqsubseteq \llbracket N'_f \rrbracket \in c$

Let $\sim \in \{<, >\}$ and let $(\gamma_1, \gamma_2) \in \mathcal{G}_i^{\sim} \llbracket \Gamma^{\sqsubseteq} \rrbracket$. We need to show

$$\left(\text{if } M\{N_t\}\{N_f\}[\gamma_1], \text{if } M'\{N'_t\}\{N'_f\}[\gamma_2] \right) \in \mathcal{E}_i^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket.$$

By Lemma D.18, it will suffice to show that (1) $(\llbracket M \rrbracket[\gamma_1], \llbracket M' \rrbracket[\gamma_2]) \in \mathcal{E}_i^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket \text{bool} \rrbracket$, and (2) for all $k \leq i$ and $(V_1, V_2) \in \mathcal{V}_k^{\sim} \llbracket \text{bool} \rrbracket$, we have

$$(\text{if } V_1\{N_t[\gamma_1]\}\{N_f[\gamma_1]\}, \text{if } V_2\{N'_t[\gamma_2]\}\{N'_f[\gamma_2]\}) \in \mathcal{E}_k^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket.$$

We note that (1) follows by our first top-level assumption. For (2), the assumption $(V_1, V_2) \in \mathcal{V}_k^{\sim} \llbracket \text{bool} \rrbracket$ has two cases. If $V_1 = V_2 = \text{true}$, then by anti-reduction (Lemma D.8), it will suffice to show $(N_t[\gamma_1], N'_t[\gamma_2]) \in \mathcal{E}_k^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket$. But this follows from our second top-level assumption. Similarly, if $V_1 = V_2 = \text{false}$, then it suffices to show that $(N_f[\gamma_1], N'_f[\gamma_2]) \in \mathcal{E}_k^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket$, which follows from our third top-level assumption. □

LEMMA D.27 (CONGRUENCE FOR LET).

PROOF. This proof is similar to the function abstraction proof and is hence omitted. □

LEMMA D.28 (CONGRUENCE FOR RAISE).

PROOF. Let $c : A_1 \sqsubseteq A_2$ and $d : B_1 \sqsubseteq B_2$. Suppose $\varepsilon@c \rightsquigarrow d \in d_\sigma$ and

$$\Gamma^{\sqsubseteq} \vDash_{d_\sigma} M_1 \sqsubseteq M_2 \in c.$$

We need to show that

$$\Gamma^{\sqsubseteq} \vDash_{d_\sigma} \text{raise } (\varepsilon@A_1 \rightsquigarrow B_1)(M_1) \sqsubseteq \text{raise } (\varepsilon@A_2 \rightsquigarrow B_2)(M_2) \in d.$$

Let $\sim \in \{<, >\}$ and $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim} \llbracket \Gamma \rrbracket$. We will show

$$(\text{raise } (\varepsilon@A_1 \rightsquigarrow B_1)(M_1)[\gamma_1], \text{raise } (\varepsilon@A_2 \rightsquigarrow B_2)(M_2)[\gamma_2]) \in \mathcal{E}_j^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket d \rrbracket.$$

We apply Lemma D.18. We first claim that $(M_1[\gamma_1], M_2[\gamma_2]) \in \mathcal{E}_j^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket$. This follows by assumption. Now, let $k \leq j$ and $(V_1, V_2) \in \mathcal{V}_k^{\sim} \llbracket c \rrbracket$. We claim that

$$(\text{raise } (\varepsilon@A_1 \rightsquigarrow B_1)(V_1)[\gamma_1], \text{raise } (\varepsilon@A_2 \rightsquigarrow B_2)(V_2)[\gamma_2]) \in \mathcal{E}_k^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket d \rrbracket.$$

By Lemma D.5, it suffices to show that

$$(\text{raise } (\varepsilon@A_1 \rightsquigarrow B_1)(V_1)[\gamma_1], \text{raise } (\varepsilon@A_2 \rightsquigarrow B_2)(V_2)[\gamma_2]) \in \mathcal{R}_k^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket d \rrbracket.$$

We assert the second disjunct in the definition of $\mathcal{R}^{\sim} \llbracket \cdot \rrbracket$, where we take ε to be ε (which we know by assumption is in d_σ), and we take $E^l = E^r = \bullet$ and $V^l = V_1, V^r = V_2$.

We need to show that $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^{\sim} \llbracket c \rrbracket)_k$, and that

$$(x^l.(\bullet[x^l]), x^r.(\bullet[x^r])) \in (\blacktriangleright \mathcal{K}^{\sim} \llbracket d \rrbracket)_k (\mathcal{E}^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket d \rrbracket)$$

To this end, let $k' \leq k$ and let $(V^l, V^r) \in \mathcal{V}_{k'}^{\sim} \llbracket c \rrbracket$. We need to show

$$(V^l, V^r) \in \mathcal{E}_{k'}^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket d \rrbracket.$$

But this follows by Lemma D.6. □

LEMMA D.29 (CONGRUENCE FOR HANDLE).

$$\frac{\begin{array}{l} M \sqsubseteq M' : d_\sigma ! c \quad y : c \vdash N \sqsubseteq N' : d_\tau ! d \\ \forall \varepsilon @ d_i \rightsquigarrow d_o \in d_\sigma. (\varepsilon \notin \text{dom}(\phi) \wedge \varepsilon \notin \text{dom}(\phi') \wedge \varepsilon : d_i \rightsquigarrow d_o \in d_\tau) \vee \\ x : d_i, k : d_o \rightarrow_{d_\tau} d \vdash \phi(\varepsilon) \sqsubseteq \phi'(\varepsilon) : d_\tau ! d \end{array}}{\text{handle } M \{ \text{ret } y.N \mid \phi \} \sqsubseteq \text{handle } M' \{ \text{ret } y.N' \mid \phi' \} : d_\tau ! d}$$

PROOF. We use Löb induction (Lemma D.16). Assume that for all $k \leq j$ and all $(\gamma_1, \gamma_2) \in (\blacktriangleright \mathcal{G}^\sim \llbracket \Gamma^\square \rrbracket)_k$ and all $(M, M') \in (\blacktriangleright \mathcal{E}^\sim \llbracket d_\sigma \rrbracket)_k (\mathcal{V}^\sim \llbracket c \rrbracket)$, we have

$$\begin{aligned} & (\text{handle } M \{ \text{ret } x.N \mid \phi \} [\gamma_1], \\ & \text{handle } M' \{ \text{ret } x'.N' \mid \phi' \} [\gamma_2]) \\ & \in (\blacktriangleright \mathcal{E}_j^\sim \llbracket d_\tau \rrbracket)_k (\mathcal{V}^\sim \llbracket d \rrbracket). \end{aligned}$$

Let $(M, M') \in \mathcal{E}_j^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket$.

Let $\sim \in \{<, >\}$ and let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^\sim \llbracket \Gamma^\square \rrbracket$. We need to show that

$$\begin{aligned} & (\text{handle } M \{ \text{ret } x.N \mid \phi \} [\gamma_1], \\ & \text{handle } M' \{ \text{ret } x'.N' \mid \phi' \} [\gamma_2]) \\ & \in \mathcal{E}_j^\sim \llbracket d_\tau \rrbracket \mathcal{V}^\sim \llbracket d \rrbracket. \end{aligned}$$

By monadic bind (Lemma D.18), it suffices to consider the following cases:

- Let $k \leq j$ and let $(V_1, V_2) \in \mathcal{V}_k^\sim \llbracket c \rrbracket$. We need to show that

$$\begin{aligned} & (\text{handle } V_1 \{ \text{ret } x.N[\gamma_1] \mid \phi[\gamma_1] \}, \\ & \text{handle } V_2 \{ \text{ret } x'.N'[\gamma_2] \mid \phi'[\gamma_2] \}) \\ & \in \mathcal{E}_j^\sim \llbracket d_\tau \rrbracket \mathcal{V}^\sim \llbracket d \rrbracket. \end{aligned}$$

By anti-reduction (Lemma D.8), it suffices to show that

$$(N[\gamma_1][V_1/x], N'[\gamma_2][V_2/x']) \in \mathcal{E}_k^\sim \llbracket d_\tau \rrbracket \mathcal{V}^\sim \llbracket d \rrbracket.$$

This follows from the premise: if we let $\gamma'_1 = \gamma_1, V_1/x$ and $\gamma'_2 = \gamma_2, V_2/x'$, then it is easily checked that $(\gamma'_1, \gamma'_2) \in \mathcal{G}_j^\sim \llbracket \Gamma^\square, x \sqsubseteq x' : c \rrbracket$. Furthermore, $N[\gamma_1][V_1/x] = N[\gamma'_1]$ and likewise for $N[\gamma_2][V_2/x']$. The premise then implies that $(N[\gamma_1][V_1/x], N'[\gamma_2][V_2/x']) \in \mathcal{E}_k^\sim \llbracket d_\tau \rrbracket \mathcal{V}^\sim \llbracket d \rrbracket$, as needed.

- Let $k \leq j$ and let $\varepsilon @ d_i \rightsquigarrow d_o \in d_\sigma$ be an effect that is caught by either handler – i.e., $\varepsilon \in \text{dom}(\phi)$ or $\varepsilon \in \text{dom}(\phi')$. By the premise, it follows that ε is in both $\text{dom}(\phi)$ and $\text{dom}(\phi')$. Let $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^\sim \llbracket c_i \rrbracket)_k$. Let $E^l \# \varepsilon$ and $E^r \# \varepsilon$ be evaluation contexts such that

$$(x^l.E^l[x^l], x^r.E^r[x^r]) \in (\blacktriangleright \mathcal{K}^\sim \llbracket d_o \rrbracket)_k (\mathcal{E}^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket).$$

We need to show that

$$\begin{aligned}
& (\text{handle } E^l[\text{raise } \varepsilon(V^l) \{\text{ret } x.N[\gamma_1] \mid \phi[\gamma_1]\}], \\
& \quad \text{handle } E^r[\text{raise } \varepsilon(V^r) \{\text{ret } x'.N'[\gamma_2] \mid \phi'[\gamma_2]\}]) \\
& \quad \in \mathcal{E}_k^\sim[\![d_\tau]\!] \mathcal{V}^\sim[\![d]\!].
\end{aligned}$$

By anti-reduction, it suffices to show that

$$\begin{aligned}
& (\phi(\varepsilon)[\gamma_1][V^l/x][(\lambda y.\text{handle } E^l[y] \{\text{ret } x.N[\gamma_1] \mid \phi[\gamma_1]\})/k], \\
& \quad \phi'(\varepsilon)[\gamma_2][V^r/x'][(\lambda y.\text{handle } E^r[y] \{\text{ret } x'.N'[\gamma_2] \mid \phi'[\gamma_2]\})/k']) \\
& \quad \in (\blacktriangleright \mathcal{E}^\sim[\![d_\tau]\!])_k (\mathcal{V}^\sim[\![d]\!]).
\end{aligned}$$

To show this, we apply the premise, as follows. Let $H_1 = \text{handle } E^l[y] \{\text{ret } x.N[\gamma_1] \mid \phi[\gamma_1]\}$ and $H_2 = \text{handle } E^r[y] \{\text{ret } x'.N'[\gamma_2] \mid \phi'[\gamma_2]\}$. Let $\gamma'_1 = \gamma_1, V^l/x_i, (\lambda y.H_1)/k_i$ and let $\gamma'_2 = \gamma_2, V^r/x'_i, (\lambda y.H_2)/k'_i$. In order to apply the premise, we must prove that $(\gamma'_1, \gamma'_2) \in \mathcal{G}_{k'}^\sim[\![\Gamma^\square, x_i \sqsubseteq x'_i : d_i, k_i \sqsubseteq k'_i : d_o \rightarrow_{d_\tau} d]\!]$.

We first need to show that $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^\sim[\![c_i]\!])_k$. This holds by assumption. We now need to show that

$$((\lambda y.H_1), (\lambda y.H_2)) \in (\blacktriangleright \mathcal{V}^\sim[\![d_o \rightarrow_{d_\tau} d]\!])_k.$$

To this end, let $k' \leq k$ and let $(V_A, V_B) \in (\blacktriangleright \mathcal{V}^\sim[\![d_o]\!])_{k'}$. We need to show that

$$\begin{aligned}
& ((\lambda y.H_1) V_A, (\lambda y.H_2) V_B) \\
& \quad \in (\blacktriangleright \mathcal{E}^\sim[\![d_\tau]\!])_{k'} (\mathcal{V}^\sim[\![d]\!])
\end{aligned}$$

By anti-reduction, it suffices to show that

$$\begin{aligned}
& (\text{handle } E^l[V_A] \{\text{ret } x.N \mid \phi\}, \text{handle } E^r[V_B] \{\text{ret } x'.N' \mid \phi'\}) \\
& \quad \in (\blacktriangleright \mathcal{E}^\sim[\![d_\tau]\!])_{k'} (\mathcal{V}^\sim[\![d]\!]).
\end{aligned}$$

By the Löb induction hypothesis, it will suffice to show that

$$(E^l[V_A], E^r[V_B]) \in (\blacktriangleright \mathcal{E}^\sim[\![d_\sigma]\!])_{k'} (\mathcal{V}^\sim[\![c]\!]).$$

Recall that by assumption, we have

$$(x^l.E^l[x^l], x^r.E^r[x^r]) \in (\blacktriangleright \mathcal{K}^\sim[\![d_o]\!])_k (\mathcal{E}^\sim[\![d_\sigma]\!]) \mathcal{V}^\sim[\![c]\!].$$

Thus, it suffices to show that $(V_A, V_B) \in (\blacktriangleright \mathcal{V}^\sim[\![d_o]\!])_{k'}$, which is precisely our assumption. \square

Note that we do not need to show soundness of the term precision congruence rules involving casts. This will follow from the soundness of the upper and lower bound rules for casts.

COROLLARY D.30 (REFLEXIVITY). *Let M be a term such that $\Sigma \mid \Gamma \mid \Delta \vdash_\sigma M : A$. We have $\Sigma \mid \Gamma^\square \vdash_\sigma M \sqsubseteq M : A$.*

PROOF. By induction on M , using the soundness of the term precision relation already proven. \square

D.1.3 Equational Rules.

LEMMA D.31 (VALUE SUBSTITUTION).

$$\frac{x_1 \sqsubseteq x_2 : c \vDash_{d_\sigma} M \equiv N : d \quad V \equiv V' : c}{M[V/x_1] \equiv N[V'/x_2]}$$

PROOF. Suppose for all j and all $(\gamma_1, \gamma_2) \in \mathcal{G}_j^\sim[\Gamma^\square, x^l \sqsubseteq x^r : c]$, that

$$(x_1.M, x_2.N) \in \mathcal{E}_j^\sim[d_\sigma]\mathcal{V}^\sim[d]$$

and

$$(x_2.N, x_1.M) \in \mathcal{E}_j^\sim[d_\sigma]\mathcal{V}^\sim[d].$$

Further suppose that for all j ,

$$(V, V') \in \mathcal{V}_j^\sim[c]$$

and

$$(V', V) \in \mathcal{V}_j^\sim[c].$$

Let j be arbitrary, and let $(\gamma_1, \gamma_2) \in \mathcal{G}^\sim[\Gamma^\square]$. We need to show

$$(M[V/x_1], N[V'/x_2]) \in \mathcal{E}_j^\sim[d_\sigma]\mathcal{V}^\sim[d]$$

and

$$(N[V'/x_2], M[V/x_1]) \in \mathcal{E}_j^\sim[d_\sigma]\mathcal{V}^\sim[d].$$

The second statement is symmetric to the first, so we show only the first.

Let $\gamma'_1 = (\gamma_1, x_1 = V)$ and let $\gamma'_2 = (\gamma_2, x_2 = V')$.

Note that we have $M[\gamma'_1] = M[\gamma_1][V/x_1]$ and $N[\gamma'_2] = N[\gamma_2][V'/x_2]$, by definition of substitution.

By our assumption, it is sufficient to show that $(\gamma'_1, \gamma'_2) \in \mathcal{G}_j^\sim[\Gamma^\square, x_1 \sqsubseteq x_2 : c]$.

For this, it suffices to show that $(\gamma'_1(x_1), \gamma'_2(x_2)) \in \mathcal{V}_j^\sim[c]$. But $\gamma'_1(x_1) = V$ and $\gamma'_2(x_2) = V'$, so we are finished. \square

LEMMA D.32 (MONAD UNIT LEFT).

$$\text{let } x = y \text{ in } N \equiv N[y/x]$$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j be arbitrary and let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^\sim[\Gamma]$. We need to show

$$(\text{let } x = y \text{ in } N, N[y/x]) \in \mathcal{E}_j^\sim[\sigma]\mathcal{V}^\sim[B].$$

Since y is a variable and hence a value, we have by the operational semantics that

$$\text{let } x = y \text{ in } N \mapsto^1 N[y/x].$$

Thus, by anti-reduction, it suffices to show that

$$(N[y/x], N[y/x]) \in \mathcal{E}_j^\sim[\sigma]\mathcal{V}^\sim[B].$$

But this follows by reflexivity (Corollary D.30).

□

LEMMA D.33 (MONAD UNIT RIGHT).

$$\text{let } x = M \text{ in } x \equiv M$$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j be arbitrary and let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim}[\Gamma]$. We need to show

$$(\text{let } x = M \text{ in } x, M) \in \mathcal{E}_j^{\sim}[\sigma]\mathcal{V}^{\sim}[\![B]\!].$$

Since x is a variable and hence a value, we have by the operational semantics that

$$\text{let } x = M \text{ in } x \mapsto^1 M[x/x].$$

By definition of substitution, $M[x/x] = M$. Thus, by anti-reduction, it suffices to show that

$$(M, M) \in \mathcal{E}_j^{\sim}[\sigma]\mathcal{V}^{\sim}[\![B]\!].$$

This follows by reflexivity (Corollary D.30). □

LEMMA D.34 (MONAD ASSOCIATIVITY).

$$\text{let } y = (\text{let } x = M \text{ in } N) \text{ in } P \equiv \text{let } x = M \text{ in let } y = N \text{ in } P$$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j be arbitrary and let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim}[\Gamma]$. We need to show

$$(\text{let } y = (\text{let } x = M \text{ in } N) \text{ in } P, \text{let } x = M \text{ in let } y = N \text{ in } P) \in \mathcal{E}_j^{\sim}[\sigma]\mathcal{V}^{\sim}[\![B]\!].$$

We apply Lemma D.18, taking $E_1 = \text{let } y = (\text{let } x = \bullet \text{ in } N) \text{ in } P$ and $E_2 = \text{let } x = \bullet \text{ in let } y = N \text{ in } P$.

We first need to show that $(M, M) \in \mathcal{E}_j^{\sim}[\sigma]\mathcal{V}^{\sim}[\![A]\!]$, which is true by reflexivity (Corollary D.30).

Now, let $k \leq j$ and $(V_1, V_2) \in \mathcal{V}_k^{\sim}[\![A]\!]$. We need to show that

$$(\text{let } y = (\text{let } x = V_1 \text{ in } N) \text{ in } P, \text{let } x = V_2 \text{ in let } y = N \text{ in } P) \in \mathcal{E}_k^{\sim}[\sigma]\mathcal{V}^{\sim}[\![B]\!].$$

According to the operational semantics, we have

$$(\text{let } x = V_1 \text{ in } N) \mapsto^1 N[V_1/x].$$

Thus,

$$\text{let } y = (\text{let } x = V_1 \text{ in } N) \text{ in } P \mapsto^1 \text{let } y = N[V_1/x] \text{ in } P.$$

Similarly, we have

$$\text{let } x = V_2 \text{ in let } y = N \text{ in } P \mapsto^1 (\text{let } y = N \text{ in } P)[V_2/x] = \text{let } y = N[V_2/x] \text{ in } P[V_2/x].$$

Note that since x does not occur in P , we have $P[V_2/x] = P$.

Now, by anti-reduction, it suffices to show

$$(\text{let } y = N[V_1/x] \text{ in } P, \text{let } y = N[V_2/x] \text{ in } P) \in \mathcal{E}_k^{\sim}[\sigma]\mathcal{V}^{\sim}[\![B]\!].$$

We again apply Lemma D.18, this time with $E_1 = \text{let } y = \bullet \text{ in } P$ and $E_2 = \text{let } y = \bullet \text{ in } P$.

We first need to show that $(N[V_1/x], N[V_2/x]) \in \mathcal{E}_k^{\sim}[\sigma]\mathcal{V}^{\sim}[[A]]$. This follows from reflexivity (Corollary D.30) and value substitution (Lemma D.31) applied to our assumption on V_1 and V_2 .

Now let $k' \leq k$ and $(V'_1, V'_2) \in \mathcal{V}_A^{\sim}[[k']]$. We need to show that

$$(\text{let } y = V'_1 \text{ in } P, \text{let } y = V'_2 \text{ in } P) \in \mathcal{E}_{k'}^{\sim}[\sigma]\mathcal{V}^{\sim}[[B]].$$

By anti-reduction, it suffices to show

$$(P[V'_1/y], P[V'_2/y]) \in \mathcal{E}_{k'}^{\sim}[\sigma]\mathcal{V}^{\sim}[[B]].$$

This again follows from reflexivity and value substitution. □

LEMMA D.35 (*η-EXPANSION FOR BOOLEANS*).

$$M[x : \text{bool}] \equiv \text{if } x \{M[\text{true}/x]\} \{M[\text{false}/x]\}$$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j be arbitrary and let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim}[\Gamma, x_1 \sqsubseteq x_2 : \text{bool}]$. We need to show

$$(M[\gamma_1], (\text{if } x \{M[\text{true}/x]\} \{M[\text{false}/x]\})[\gamma_2]) \in \mathcal{E}_j^{\sim}[\sigma]\mathcal{V}^{\sim}[[B]].$$

By definition of substitution, this is equivalent to

$$(M[\gamma_1], (\text{if } \gamma \{ \{2\}(x)M[\text{true}/x][\gamma_2]M[\text{false}/x][\gamma_2] \})) \in \mathcal{E}_j^{\sim}[\sigma]\mathcal{V}^{\sim}[[B]].$$

By our assumption on γ_1 and γ_2 , we have that either $\gamma_1(x_1) = \gamma_2(x_2) = \text{true}$ or $\gamma_1(x_1) = \gamma_2(x_2) = \text{false}$.

We show only the former case; the latter is symmetric. In the former case, we need to show

$$(M[\text{true}/x][\gamma_1], (\text{if } \text{true} \{M[\text{true}/x][\gamma_2]\} \{M[\text{false}/x][\gamma_2]\})) \in \mathcal{E}_j^{\sim}[\sigma]\mathcal{V}^{\sim}[[B]].$$

By anti-reduction, it is sufficient to show

$$(M[\text{true}/x][\gamma_1], M[\text{true}/x][\gamma_2]) \in \mathcal{E}_j^{\sim}[\sigma]\mathcal{V}^{\sim}[[B]].$$

This follows by reflexivity. □

LEMMA D.36 (*BOOLEAN β REDUCTION - TRUE*).

$$\text{if } \text{true} \{N_t\} \{N_f\} \equiv N_t$$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j be arbitrary and let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim}[\Gamma]$. We need to show

$$((\text{if } \text{true} \{N_t\} \{N_f\})[\gamma_1], N_t[\gamma_2]) \in \mathcal{E}_j^{\sim}[\sigma]\mathcal{V}^{\sim}[[B]].$$

By anti-reduction, it suffices to show

$$(N_t[\gamma_1], N_t[\gamma_2]) \in \mathcal{E}_j^{\sim}[\sigma]\mathcal{V}^{\sim}[[B]].$$

This holds by reflexivity. □

LEMMA D.37 (*BOOLEAN β REDUCTION - FALSE*).

$$\text{if } \text{false} \{N_t\} \{N_f\} \equiv N_f$$

PROOF. Precisely dual to the above proof. \square

LEMMA D.38 (EVAL FOR IF).

$$\text{if } M\{N_t\}\{N_f\} \equiv \text{let } x = M \text{ in if } x\{N_t\}\{N_f\} \text{ }_{\text{IF EVAL}}$$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j be arbitrary and let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^\sim[\Gamma]$. We need to show

$$((\text{if } M\{N_t\}\{N_f\}[\gamma_1], (\text{let } x = M \text{ in if } x\{N_t\}\{N_f\}[\gamma_2])) \in \mathcal{E}_j^\sim[\sigma]\mathcal{V}^\sim[B].$$

We apply Lemma D.18, with $E_1 = \text{if } \bullet\{N_t[\gamma_1]\}\{N_f[\gamma_1]\}$ and $E_2 = \text{let } x = \bullet \text{ in if } \gamma_2(x)\{N_t[\gamma_2]\}\{N_f[\gamma_2]\}$.

We first need to show that $(M[\gamma_1], M[\gamma_2]) \in \mathcal{E}_j^\sim[\tau]\mathcal{V}^\sim[\text{bool}]$. This follows by reflexivity (Corollary D.30).

Now let $k \leq j$ and let $(V_1, V_2) \in \mathcal{V}_k^\sim[\text{bool}]$. We need to show that

$$((\text{if } V_1\{N_t[\gamma_1]\}\{N_f[\gamma_1]\}), (\text{let } x = V_2 \text{ in if } \gamma_2(x)\{N_t[\gamma_2]\}\{N_f[\gamma_2]\})) \in \mathcal{E}_k^\sim[\sigma]\mathcal{V}^\sim[B].$$

By definition of $\mathcal{V}^\sim[\text{bool}]$, either $V_1 = V_2 = \text{true}$ or $V_1 = V_2 = \text{false}$. We consider the first case; the second is symmetric.

We need to show

$$((\text{if } \text{true}\{N_t[\gamma_1]\}\{N_f[\gamma_1]\}), (\text{let } x = \text{true} \text{ in if } \gamma_2(x)\{N_t[\gamma_2]\}\{N_f[\gamma_2]\})) \in \mathcal{E}_k^\sim[\sigma]\mathcal{V}^\sim[B].$$

By anti-reduction, it suffices to show

$$(N_t[\gamma_1], N_t[\gamma_2]) \in \mathcal{E}_k^\sim[\sigma]\mathcal{V}^\sim[B].$$

This follows by reflexivity. \square

LEMMA D.39 (β -REDUCTION FOR FUNCTIONS).

$$(\lambda x.M)V \equiv M[V/x] \text{ }_{\text{FUNBETA}}$$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j be arbitrary and let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^\sim[\Gamma]$. We need to show

$$(((\lambda x.M)V)[\gamma_1], (M[V/x])[\gamma_2]) \in \mathcal{E}_j^\sim[\sigma]\mathcal{V}^\sim[B].$$

Since V is a value, it suffices by anti-reduction to show that

$$(M[V/x][\gamma_1], M[V/x][\gamma_2]) \in \mathcal{E}_j^\sim[\sigma]\mathcal{V}^\sim[B].$$

This follows by reflexivity. \square

LEMMA D.40 (η -EXPANSION FOR FUNCTIONS). Let V_f be a value such that $\Sigma \mid \Gamma \mid \Delta \vdash_{\emptyset} V : A \rightarrow_{\sigma'} B$. We have $\Sigma \mid \Gamma^\square \vDash_{\sigma} V_f \equiv (\lambda x.V_fx) : (A \rightarrow_{\sigma'} B)$.

PROOF. Let j be arbitrary. We need to show

$$(V_f, (\lambda x.V_fx)) \in \mathcal{E}_j^\sim[\emptyset]\mathcal{V}^\sim[A \rightarrow_{\sigma'} B].$$

As these are values, it suffices by Lemma D.6 to show that they are related in $\mathcal{V}_j^\sim[A \rightarrow_{\sigma'} B]$. To this end, let $k \leq j$ and let $(V_{i1}, V_{i2}) \in \mathcal{V}_k^\sim[A]$. We claim that

$$(V_f V_{i1}, (\lambda x. V_f x) V_{i2}) \in \mathcal{E}_k^{\sim}[\![\sigma']\!] \mathcal{V}^{\sim}[\![B]\!].$$

By anti-reduction, it will suffice to show that

$$(V_f V_{i1}, V_f V_{i2}) \in \mathcal{E}_k^{\sim}[\![\sigma']\!] \mathcal{V}^{\sim}[\![B]\!].$$

By reflexivity (Corollary D.30), we know that $(V_f, V_f) \in \mathcal{E}_k^{\sim}[\![\emptyset]\!] A \rightarrow_{\sigma'} B$, and since V_f is a value, this means that $(V_f, V_f) \in \mathcal{V}_k^{\sim}[\![A \rightarrow_{\sigma'} B]\!]$. This immediately implies the desired result, since $(V_{i1}, V_{i2}) \in \mathcal{V}_k^{\sim}[\![A]\!]$. \square

LEMMA D.41 (APPEVAL).

$$M N \equiv \text{let } x = M \text{ in let } y = N \text{ in } x y$$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j be arbitrary and let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim}[\![\Gamma]\!]$. We need to show

$$((M N)[\gamma_1], (\text{let } x = M \text{ in let } y = N \text{ in } x y)[\gamma_2]) \in \mathcal{E}_j^{\sim}[\![\tau_A]\!] \mathcal{V}^{\sim}[\![A_o]\!].$$

We apply Lemma D.18, with $E_1 = (\bullet N[\gamma_2])$ and $E_2 = \text{let } x = \bullet \text{ in let } y = N[\gamma_2] \text{ in } x y$.

We first need to show that $(M[\gamma_1], M[\gamma_2]) \in \mathcal{E}_j^{\sim}[\![\tau]\!] \mathcal{V}^{\sim}[\![A_i \rightarrow_{\tau_A} A_o]\!]$. This follows by reflexivity.

Now let $k \leq j$ and let $(V_1, V_2) \in \mathcal{V}_k^{\sim}[\![A_i \rightarrow_{\sigma_A} A_o]\!]$. We need to show that

$$((V_1 N[\gamma_1]), (\text{let } x = V_2 \text{ in let } y = N[\gamma_2] \text{ in } x y)) \in \mathcal{E}_k^{\sim}[\![\tau_A]\!] \mathcal{V}^{\sim}[\![A_o]\!].$$

By anti-reduction, it suffices to show

$$((V_1 N[\gamma_1]), (\text{let } y = N[\gamma_2] \text{ in } V_2 y)) \in \mathcal{E}_k^{\sim}[\![\tau_A]\!] \mathcal{V}^{\sim}[\![A_o]\!].$$

We again apply Lemma D.18, this time with $E_1 = (V_1 \bullet)$ and $E_2 = \text{let } y = \bullet \text{ in } V_2 y$.

We need to show $(N[\gamma_1], N[\gamma_2]) \in \mathcal{E}_k^{\sim}[\![\tau]\!] \mathcal{V}^{\sim}[\![A_i]\!]$, which holds by reflexivity. Now let $k' \leq k$ and let $(V'_1, V'_2) \in \mathcal{V}_{k'}^{\sim}[\![A_i]\!]$. We need to show that

$$((V_1 V'_1), (\text{let } y = V'_2 \text{ in } V_2 y)) \in \mathcal{E}_{k'}^{\sim}[\![\tau_A]\!] \mathcal{V}^{\sim}[\![A_o]\!].$$

By anti-reduction, it suffices to show

$$((V_1 V'_1), (V_2 V'_2)) \in \mathcal{E}_{k'}^{\sim}[\![\tau_A]\!] \mathcal{V}^{\sim}[\![A_o]\!].$$

This follows from our assumptions on V_1 and V_2 and on V'_1 and V'_2 . \square

LEMMA D.42 (HANDLEBETARET).

$$\text{handle } x \{ \text{ret } y.M \mid \phi \} \equiv M[x/y]$$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j be arbitrary and let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim}[\![\Gamma]\!]$. We need to show

$$((\text{handle } x \{ \text{ret } y.M \mid \phi \})[\gamma_1], (M[x/y])[\gamma_2]) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![B]\!].$$

Since x is a value, the above handle term steps, and by anti-reduction it is sufficient to show

$$((M[x/y][\gamma_1]), (M[x/y])[\gamma_2]) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![B]\!].$$

This follows by reflexivity.

□

LEMMA D.43 (HANDLEBETARAISE).

$\text{handle } (\text{let } o = \text{raise } \varepsilon(x) \text{ in } N_k) \{ \text{ret } y.M \mid \phi \} \equiv \phi(\varepsilon)[\lambda o.\text{handle } N_k \{ \text{ret } y.M \mid \phi \}/k]$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j be arbitrary and let $(\gamma_1, \gamma_2) \in \mathcal{E}_j^\sim[\Gamma]$. We need to show

$$\begin{aligned} & ((\text{handle } (\text{let } o = \text{raise } \varepsilon(x) \text{ in } N_k) \{ \text{ret } y.M \mid \phi \})[\gamma_1], \\ & (\phi(\varepsilon)[\lambda o.\text{handle } N_k \{ \text{ret } y.M \mid \phi \}/k])[\gamma_2]) \\ & \in \mathcal{E}_j^\sim[\sigma]\mathcal{V}^\sim[B]. \end{aligned}$$

Let $E = \text{let } o = \bullet \text{ in } N_k[\gamma_1]$. Our goal is to show

$$\begin{aligned} & ((\text{handle } E[\text{raise } \varepsilon(x)] \{ \text{ret } y.M[\gamma_1] \mid \phi[\gamma_1] \}), \\ & (\phi(\varepsilon)[\gamma_2][\lambda o.\text{handle } N_k[\gamma_2] \{ \text{ret } y.M[\gamma_2] \mid \phi[\gamma_2] \}/k])) \\ & \in \mathcal{E}_j^\sim[\sigma]\mathcal{V}^\sim[B]. \end{aligned}$$

Note that $E \# \varepsilon$. By anti-reduction, it suffices to show

$$\begin{aligned} & ((\phi(\varepsilon)[\gamma_1][\lambda o'.\text{handle } E[o'] \{ \text{ret } y.M[\gamma_1] \mid \phi[\gamma_1] \}/k]), \\ & (\phi(\varepsilon)[\gamma_2][\lambda o.\text{handle } N_k[\gamma_2] \{ \text{ret } y.M[\gamma_2] \mid \phi[\gamma_2] \}/k])) \\ & \in \mathcal{E}_j^\sim[\sigma]\mathcal{V}^\sim[B]. \end{aligned}$$

That is, we need to show

$$\begin{aligned} & ((\phi(\varepsilon)[\gamma_1][\lambda o'.\text{handle } \text{let } o = o' \text{ in } N_k[\gamma_1] \{ \text{ret } y.M[\gamma_1] \mid \phi[\gamma_1] \}/k]), \\ & (\phi(\varepsilon)[\gamma_2][\lambda o.\text{handle } N_k[\gamma_2] \{ \text{ret } y.M[\gamma_2] \mid \phi[\gamma_2] \}/k])) \\ & \in \mathcal{E}_j^\sim[\sigma]\mathcal{V}^\sim[B]. \end{aligned}$$

By ValSubst, it suffices to show (1) for all related $(V_{f_1}, V_{f_2}) \in \mathcal{V}_j^\sim[A_i \rightarrow_\sigma B]$ and $\gamma'_1 = \gamma_1, V_{f_1}/k$ and $\gamma'_2 = \gamma_2, V_{f_2}/k$, we have

$$(\phi(\varepsilon)[\gamma'_1], \phi(\varepsilon)[\gamma'_2]) \in \mathcal{E}_j^\sim[\sigma]\mathcal{V}^\sim[B],$$

and (2),

$$\begin{aligned} & ((\lambda o'.\text{handle } \text{let } o = o' \text{ in } N_k[\gamma_1] \{ \text{ret } y.M[\gamma_1] \mid \phi[\gamma_1] \}), \\ & (\lambda o.\text{handle } N_k[\gamma_2] \{ \text{ret } y.M[\gamma_2] \mid \phi[\gamma_2] \})) \\ & \in \mathcal{E}_j^\sim[\sigma]\mathcal{V}^\sim[A_i \rightarrow_\sigma B]. \end{aligned}$$

(1) follows from reflexivity. To show (2), we will use transitivity (Lemma D.66). If \sim is $<$, then note that by MonadUnitL we have

$$(\text{let } o = o' \text{ in } N_k[\gamma_1], N_k[\gamma_2][o'/o]) \in \mathcal{E}_j^\sim[\sigma]\mathcal{V}^\sim[B],$$

and by soundness of the congruence rules we have

$$\begin{aligned}
& ((\lambda o'. \text{handle } \text{let } o = o' \text{ in } N_k[\gamma_1] \{ \text{ret } y.M[\gamma_1] \mid \phi[\gamma_1] \}), \\
& (\lambda o'. \text{handle } N_k[\gamma_2][o'/o] \{ \text{ret } y.M[\gamma_2] \mid \phi[\gamma_2] \})) \\
& \in \mathcal{E}_j^\sim[\sigma] \mathcal{V}^\sim[A_i \rightarrow_\sigma B].
\end{aligned}$$

Then by transitivity, it will suffice to show that

$$\begin{aligned}
& ((\lambda o'. \text{handle } N_k[\gamma_2][o'/o] \{ \text{ret } y.M[\gamma_2] \mid \phi[\gamma_2] \}), \\
& (\lambda o. \text{handle } N_k[\gamma_2] \{ \text{ret } y.M[\gamma_2] \mid \phi[\gamma_2] \})) \\
& \in \mathcal{E}_\omega^\sim[\sigma] \mathcal{V}^\sim[A_i \rightarrow_\sigma B].
\end{aligned}$$

By congruence for lambdas, it suffices to show that, given related values $(V_1, V_2) \in \mathcal{V}_\omega^\sim[A_i]$, we have

$$\begin{aligned}
& ((\text{handle } N_k[\gamma_2][o'/o][V_1/o'] \{ \text{ret } y.M[\gamma_2] \mid \phi[\gamma_2] \}), \\
& (\text{handle } N_k[\gamma_2][V_2/o] \{ \text{ret } y.M[\gamma_2] \mid \phi[\gamma_2] \})) \\
& \in \mathcal{E}_\omega^\sim[\sigma] \mathcal{V}^\sim[A_i \rightarrow_\sigma B].
\end{aligned}$$

This follows from the soundness of the congruence rules.

On the other hand, if \sim is $>$, then similarly by `MonadUnitL` we have

$$(\text{let } o = o' \text{ in } N_k[\gamma_1], N_k[\gamma_1][o'/o]) \in \mathcal{E}_\omega^\sim[\sigma] \mathcal{V}^\sim[B].$$

It then suffices to show that

$$\begin{aligned}
& ((\lambda o'. \text{handle } N_k[\gamma_1][o'/o] \{ \text{ret } y.M[\gamma_1] \mid \phi[\gamma_1] \}), \\
& (\lambda o. \text{handle } N_k[\gamma_2] \{ \text{ret } y.M[\gamma_2] \mid \phi[\gamma_2] \})) \\
& \in \mathcal{E}_j^\sim[\sigma] \mathcal{V}^\sim[A_i \rightarrow_\sigma B],
\end{aligned}$$

which again follows from the soundness of the congruence rules. \square

LEMMA D.44 (RAISEEVAL).

$$\text{raise } \varepsilon(M) \equiv \text{let } x = M \text{ in raise } \varepsilon(x)$$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j be arbitrary and let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^\sim[\Gamma]$. We need to show

$$((\text{raise } \varepsilon(M))[\gamma_1], (\text{let } x = M \text{ in raise } \varepsilon(x))[\gamma_2]) \in \mathcal{E}_j^\sim[\sigma] \mathcal{V}^\sim[B].$$

We apply `Monadic Bind` (Lemma D.18), with $E_1 = \text{raise } \varepsilon(\bullet)$ and $E_2 = \text{let } x = \bullet \text{ in raise } \varepsilon(x)$.

We first need to show that $(M[\gamma_1], M[\gamma_2]) \in \mathcal{E}_j^\sim[\tau] \mathcal{V}^\sim[A]$. This follows from reflexivity (Corollary D.30).

Now let $k \leq j$ and let $(V_1, V_2) \in \mathcal{V}_k^\sim[A]$. We need to show that

$$((\text{raise } \varepsilon(V_1))[\gamma_1], (\text{let } x = V_2 \text{ in raise } \varepsilon(x))[\gamma_2]) \in \mathcal{E}_k^\sim[\sigma] \mathcal{V}^\sim[B].$$

As V_2 is a value, the above `let` term steps. By anti-reduction, it suffices to show

$$((\text{raise } \varepsilon(V_1)), (\text{raise } \varepsilon(V_2))) \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![B]\!].$$

This follows from our assumption on V_1 and V_2 and the soundness of the term congruence rule for `raise` (Lemma D.28). □

LEMMA D.45 (HANDLEEMPTY).

$$\text{handle } M \{\text{ret } x.N \mid \emptyset\} \equiv \text{let } x = M \text{ in } N$$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j be arbitrary and let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim}[\![\Gamma]\!]$. We need to show

$$\begin{aligned} & ((\text{handle } M \{\text{ret } x.N \mid \emptyset\})[\gamma_1], \\ & (\text{let } x = M \text{ in } N)[\gamma_2]) \\ & \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![B]\!]. \end{aligned}$$

By Monadic Bind (Lemma D.18) and the fact that neither evaluation context catches any effects, it suffices to show that

$$\begin{aligned} & (\text{handle } V_1 \{\text{ret } x.N[\gamma_1] \mid \emptyset\}, \\ & \text{let } x = V_2 \text{ in } N[\gamma_2]) \\ & \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![B]\!], \end{aligned}$$

where $k \leq j$ and $(V_1, V_2) \in \mathcal{V}_k^{\sim}[\![B]\!]$. By anti-reduction, it will suffice to show that

$$(N[\gamma_1][V_1/x], N[\gamma_2][V_2/x]) \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![B]\!].$$

Using ValSubst, the result follows by reflexivity and our assumption on V_1 and V_2 . □

LEMMA D.46.

$$\frac{\forall \varepsilon \in \text{dom}(\phi). \psi(\varepsilon) = \phi(\varepsilon) \quad \forall \varepsilon \in \text{dom}(\psi). \varepsilon \notin \text{dom}(\phi) \Rightarrow \psi(\varepsilon) = k(\text{raise } \varepsilon(x))}{\text{handle } M \{\text{ret } y.N \mid \phi\} \equiv \text{handle } M \{\text{ret } y.N \mid \psi\} : \sigma ! B} \text{HANDLEEXT}$$

PROOF. We show one direction of the equivalence; the other is symmetric. The proof is by Löb induction. We assume that

$$((\text{handle } M \{\text{ret } .1 \mid 'yN\phi\})[\gamma_1], (\text{handle } M \{\text{ret } .2 \mid 'yN\psi\})[\gamma_2]) \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma]\!])_j(\mathcal{V}^{\sim}[\![B]\!]).$$

for all $k \leq j$, $(\gamma_1, \gamma_2) \in (\blacktriangleright \mathcal{G}^{\sim}[\![\Gamma]\!])_k$ and $(M'_1, M'_2) \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma]\!])_k(\mathcal{V}^{\sim}[\![A]\!])$.

Let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim}[\![\Gamma]\!]$. We need to show

$$((\text{handle } M \{\text{ret } .1 \mid y\}N\phi)[\gamma_1], (\text{handle } M \{\text{ret } .2 \mid y\}N\psi)[\gamma_2]) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![B]\!]$$

for all $(M_1, M_2) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!]$.

We apply Monadic Bind (Lemma D.18). It suffices to consider the following cases:

- Let $k \leq j$ and $(V_1, V_2) \in \mathcal{V}_k^{\sim} \llbracket A \rrbracket$. We need to show that

$$((\text{handle } V_1 \{ \text{ret } y.N[\gamma_1] \mid \phi[\gamma_1] \}), (\text{handle } V_2 \{ \text{ret } y.N[\gamma_2] \mid \psi[\gamma_2] \})) \in \mathcal{E}_k^{\sim} \llbracket \sigma \rrbracket \mathcal{V}^{\sim} \llbracket B \rrbracket.$$

This follows by anti-reduction and reflexivity.

- Let $k \leq j$ and let $\varepsilon \in \sigma$ be an effect caught by either handler, i.e., ε is in $\text{dom}(\phi)$ or $\text{dom}(\psi)$. Let $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^{\sim} \llbracket c_\varepsilon \rrbracket)_k$, and let $E^l \# \varepsilon$ and $E^r \# \varepsilon$ such that $(x^l.E^l[x^l], x^r.E^r[x^r]) \in (\blacktriangleright \mathcal{K}^{\sim} \llbracket d_\varepsilon \rrbracket)_k (\mathcal{E}^{\sim} \llbracket \sigma \rrbracket \mathcal{V}^{\sim} \llbracket B \rrbracket)$.

We need to show

$$\begin{aligned} & ((\text{handle } E^l[\text{raise } \varepsilon(V^l)] \{ \text{ret } y.N[\gamma_1] \mid \phi[\gamma_1] \}), \\ & (\text{handle } E^r[\text{raise } \varepsilon(V^r)] \{ \text{ret } y.N[\gamma_2] \mid \psi[\gamma_2] \})) \\ & \in \mathcal{E}_k^{\sim} \llbracket \sigma \rrbracket \mathcal{V}^{\sim} \llbracket B \rrbracket. \end{aligned}$$

If $\varepsilon \in \text{dom}(\phi)$, then by the premise, we have $\psi(\varepsilon) = \phi(\varepsilon)$, so both sides step, and it suffices by anti-reduction to show

$$\begin{aligned} & (\phi(\varepsilon)[\gamma_1][V^l/x][(\lambda z.\text{handle } E^l[z] \{ \text{ret } y.N[\gamma_1] \mid \phi[\gamma_1] \})/k], \\ & \phi(\varepsilon)[\gamma_2][V^r/x][(\lambda z.\text{handle } E^r[z] \{ \text{ret } y.N[\gamma_2] \mid \psi[\gamma_2] \})/k]) \\ & \in (\blacktriangleright \mathcal{E}^{\sim} \llbracket \sigma \rrbracket)_k (\mathcal{V}^{\sim} \llbracket B \rrbracket). \end{aligned}$$

By ValSubst, it suffices to show that $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^{\sim} \llbracket c_\varepsilon \rrbracket)_k$, which is true by assumption, and that

$$\begin{aligned} & ((\lambda z.\text{handle } E^l[z] \{ \text{ret } y.N[\gamma_1] \mid \phi[\gamma_1] \}), \\ & (\lambda z.\text{handle } E^r[z] \{ \text{ret } y.N[\gamma_2] \mid \psi[\gamma_2] \})) \\ & \in (\blacktriangleright \mathcal{V}^{\sim} \llbracket d_\varepsilon \rightarrow_\sigma B \rrbracket)_k. \end{aligned}$$

By congruence for lambdas, it suffices to show that, given values $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^{\sim} \llbracket d_\varepsilon \rrbracket)_k$, we have

$$\begin{aligned} & (\text{handle } E^l[V_1] \{ \text{ret } y.N[\gamma_1] \mid \phi[\gamma_1] \}), \\ & \text{handle } E^r[V_2] \{ \text{ret } y.N[\gamma_2] \mid \psi[\gamma_2] \}) \\ & \in (\blacktriangleright \mathcal{E}^{\sim} \llbracket \sigma \rrbracket)_k (\mathcal{V}^{\sim} \llbracket B \rrbracket). \end{aligned}$$

This follows by the Löb induction hypothesis and our assumption on E^l and E^r .

Now assume that $\varepsilon \notin \text{dom}(\phi)$. Then note that the first handle term does not step, while the second handle term steps to

$$\psi(\varepsilon)[\gamma_2][V^r/x][(\lambda z.\text{handle } E^r[z] \{ \text{ret } y.N[\gamma_2] \mid \psi[\gamma_2] \})/k].$$

By the premise, we have $\psi(\varepsilon) = k(\text{raise } \varepsilon(x))$. Thus, by anti-reduction, it suffices to show

$$\begin{aligned} & ((\text{handle } E^l[\text{raise } \varepsilon(V^l)] \{ \text{ret } y.N[\gamma_1] \mid \phi[\gamma_1] \}), \\ & (k(\text{raise } \varepsilon(x))[\gamma_2][V^r/x][(\lambda z.\text{handle } E^r[z] \{ \text{ret } y.N[\gamma_2] \mid \psi[\gamma_2] \})/k]) \\ & \in \mathcal{E}_k^{\sim} \llbracket \sigma \rrbracket \mathcal{V}^{\sim} \llbracket B \rrbracket. \end{aligned}$$

That is, it will suffice to show

$$\begin{aligned} & ((\text{handle } E^l[\text{raise } \varepsilon(V^l)] \{ \text{ret } y.N[\gamma_1] \mid \phi[\gamma_1] \})), \\ & ((\lambda z.\text{handle } E^r[z] \{ \text{ret } y.N[\gamma_2] \mid \psi[\gamma_2] \}) (\text{raise } \varepsilon(V^r))) \\ & \in \mathcal{E}_k^{\sim}[\sigma]\mathcal{V}^{\sim}[\![B]\!]. \end{aligned}$$

Neither term steps, so it suffices to show they are related in $\mathcal{R}_k^{\sim}[\sigma]\mathcal{V}^{\sim}[\![B]\!]$.

We need to show that $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^{\sim}[\![c_\varepsilon]\!])_k$, which is true by assumption, and that given $k' \leq k$ and related values $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^{\sim}[\![d_\varepsilon]\!])_{k'}$, we have

$$\begin{aligned} & ((\text{handle } E^l[V_1] \{ \text{ret } y.N[\gamma_1] \mid \phi[\gamma_1] \})), \\ & ((\lambda z.\text{handle } E^r[z] \{ \text{ret } y.N[\gamma_2] \mid \psi[\gamma_2] \}) V_2)) \\ & \in (\blacktriangleright \mathcal{E}^{\sim}[\sigma])_{k'}(\mathcal{V}^{\sim}[\![B]\!]). \end{aligned}$$

By anti-reduction, it suffices to show

$$\begin{aligned} & ((\text{handle } E^l[V_1] \{ \text{ret } y.N[\gamma_1] \mid \phi[\gamma_1] \})), \\ & (\text{handle } E^r[V_2] \{ \text{ret } y.N[\gamma_2] \mid \psi[\gamma_2] \})) \\ & \in (\blacktriangleright \mathcal{E}^{\sim}[\sigma])_{k'}(\mathcal{V}^{\sim}[\![B]\!]). \end{aligned}$$

This follows by the Löb induction hypothesis and our assumption on E^l and E^r . □

D.1.4 Cast, Error, and Subtyping Properties.

LEMMA D.47 (ERR-BOT).

$$\frac{M : d_\sigma^r ! c^r}{\mathcal{U} \sqsubseteq M : d_\sigma ! c}$$

PROOF. Let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim}[\![\Gamma^E]\!]$. We need to show

$$(\mathcal{U}[\gamma_1], M[\gamma_2]) \in \mathcal{E}_j^{\sim}[\![d_\sigma]\!]\mathcal{V}^{\sim}[\![c]\!].$$

This follows from the definition of the logical relation: If \sim is $<$ (counting steps on the left), then we are finished by the definition of the $\mathcal{E}^{\leq}[\![\]\!]$ relation, because $\mathcal{U} \mapsto^0 \mathcal{U}$.

If \sim is $>$ (counting steps on the right), then we are similarly finished, because $M \mapsto^0 M$ and the left-hand term is \mathcal{U} . □

LEMMA D.48 (ERR-STRICT). $E[\mathcal{U}] \equiv \mathcal{U}$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j, d_σ , and c be arbitrary. We need to show

$$(E[\mathcal{U}], \mathcal{U}) \in \mathcal{E}_j^{\sim}[\![d_\sigma]\!]\mathcal{V}^{\sim}[\![c]\!].$$

By anti-reduction, it is sufficient to show

$$(\mathcal{U}, \mathcal{U}) \in \mathcal{E}_j^{\sim}[\![d_\sigma]\!]\mathcal{V}^{\sim}[\![c]\!],$$

which is easily seen to hold by definition of the logical relation.

□

LEMMA D.49 (MONOTONICITY OF SUBTYPING). *If $c \leq d$ then $\mathcal{V}[\![c]\!] \subseteq \mathcal{V}[\![d]\!]$*

Further, if $R \subseteq S$ then $\mathcal{K}[\![d]\!]R \subseteq \mathcal{K}[\![c]\!]S$,

Further, if $c_\sigma \leq d_\sigma$ then both

- $\mathcal{E}[\![c]\!]R \subseteq \mathcal{E}[\![d]\!]S$
- $\mathcal{R}[\![c]\!]R \subseteq \mathcal{R}[\![d]\!]S$

PROOF. By mutual induction on the subtyping proofs. First the type subtyping cases:

- (1) $\text{bool} \leq \text{bool}$: trivial.
- (2) $c_i \rightarrow_{c_e} c_o \leq d_i \rightarrow_{d_e} d_o$. Assume $(V_f, V'_f) \in \mathcal{V}[\![c_i \rightarrow_{c_e} c_o]\!]$, we need to show $(V_f, V'_f) \in \mathcal{V}[\![d_i \rightarrow_{d_e} d_o]\!]$. Let $(V_i, V'_i) \in \mathcal{V}[\![d_i]\!]$. Then by inductive hypothesis, $(V_i, V'_i) \in \mathcal{V}[\![c_i]\!]$. Therefore $(V_f V_i, V'_f V'_i) \in \mathcal{E}[\![c_e]\!]\mathcal{V}[\![c_o]\!]$ and the result follows by the two inductive hypotheses.

The $\mathcal{K}[\![\cdot]\!]$ case follows by a similar argument to the function case.

The $\mathcal{E}[\![\cdot]\!]$ case follows by inductive hypothesis.

Next the $\mathcal{R}[\![\cdot]\!]$ cases:

- (1) $? \leq ?$: trivial
- (2) $\frac{c \leq \Sigma}{c \leq ?}$: trivial by definition of $\mathcal{R}[\![?]\!]$
- (3) $\frac{c \leq d}{c \leq \text{Inj}(d)}$: trivial by definition of $\mathcal{R}[\![\text{Inj}(i, d)]\!]$
- (4) $\frac{c \leq d}{\text{Inj}(c) \leq \text{Inj}(d)}$: trivial by definition of $\mathcal{R}[\![\text{Inj}(i, d)]\!]$
- (5) $\frac{\text{dom}(d_c) \subseteq \text{dom}(d'_c) \quad \forall \varepsilon : c \rightsquigarrow d \in d_c. \varepsilon : c' \rightsquigarrow d' \in d'_c \wedge c \leq c' \wedge d' \leq d}{d_c \leq d'_c}$: Follows using Löb induction by the monotonicity of subtyping for the $\mathcal{V}^\sim[\![\cdot]\!]$ and $\mathcal{K}^\sim[\![\cdot]\!]$ relations.

□

We next prove generalized versions of the cast properties ValUpL, ValUpR, ValDnL, ValDnR, EffUpL, EffUpR, EffDnL, EffDnR. These are proved simultaneously by induction on the type precision derivation and by Löb-induction.

LEMMA D.50 (VALUPR-GENERAL).

$$\frac{\begin{array}{l} c : A \sqsubseteq A' \\ e : A' \sqsubseteq A'' \\ \Sigma \mid \Gamma^\square \vDash_{d_\sigma} M \sqsubseteq N : c \end{array}}{\Sigma \mid \Gamma^\square \vDash_{d_\sigma} M \sqsubseteq \langle A'' \leftarrow A' \rangle N : c \circ e}$$

PROOF. We need to show that

$$(M, \langle A'' \leftarrow A' \rangle N) \in \mathcal{E}_j^\sim[\![d_\sigma]\!]\mathcal{V}^\sim[\![c \circ e]\!].$$

The proof is by induction on the precision derivation e . By monadic bind (Lemma D.18), with $E_1 = \bullet$ and $E_2 = \langle A'' \leftarrow A' \rangle \bullet$, it suffices to show

$$(V_1, \langle A'' \leftarrow A' \rangle V_2) \in \mathcal{E}_k^\sim[\![d_\sigma]\!]\mathcal{V}^\sim[\![c \circ e]\!],$$

where $k \leq j$ and $(V_1, V_2) \in \mathcal{V}_k^\sim \llbracket c \rrbracket$. We continue by cases on e .

- Case $e = \text{bool}$. We have $A = A' = A'' = \text{bool}$, and $c = \text{bool}$. Thus $c \circ e = \text{bool}$. Examining the operational semantics, we see that

$$((\text{bool} \prec_{\mathcal{V}} \text{bool}))(V_1) \mapsto^1 V_1.$$

Thus, by anti-reduction, it suffices to show

$$(V_1, V_2) \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket \text{bool} \rrbracket.$$

This is true by assumption and Lemma D.6.

- Case $e = e_i \rightarrow_{e_\sigma} e_o$. We have $A' = A'_i \rightarrow_{\sigma'_A} A'_o$ and $A'' = A''_i \rightarrow_{\sigma''_A} A''_o$, and also $e_i : A'_i \sqsubseteq A'_i$ and $e_o : A'_o \sqsubseteq A''_o$.

By inversion, we see that $c = c_i \rightarrow_{c_\sigma} c_o$. Thus, we have that $c \circ e = (c_i \rightarrow_{c_\sigma} c_o) \circ (e_i \rightarrow_{e_\sigma} e_o) = (c_i \circ e_i) \rightarrow_{c_\sigma \circ e_\sigma} (c_o \circ e_o)$.

We need to show that

$$(V_1, \langle (A''_i \rightarrow_{\sigma''_A} A''_o) \prec_{\mathcal{V}} (A'_i \rightarrow_{\sigma'_A} A'_o) \rangle V_2) \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket (c_i \circ e_i) \rightarrow_{c_\sigma \circ e_\sigma} (c_o \circ e_o) \rrbracket.$$

As both terms are values, it suffices by Lemma D.6 to show they are related in $\mathcal{V}_k^\sim \llbracket (c_i \circ e_i) \rightarrow_{c_\sigma \circ e_\sigma} (c_o \circ e_o) \rrbracket$.

To this end, let $k' \leq k$ and $(V^l, V^r) \in \mathcal{V}_{k'}^\sim \llbracket c_i \circ e_i \rrbracket$. We need to show that

$$(V_1 V^l, \langle (A''_i \rightarrow_{\sigma''_A} A''_o) \prec_{\mathcal{V}} (A'_i \rightarrow_{\sigma'_A} A'_o) \rangle V_2 V^r) \in \mathcal{E}_{k'}^\sim \llbracket c_\sigma \circ e_\sigma \rrbracket \mathcal{V}^\sim \llbracket c_o \circ e_o \rrbracket.$$

By anti-reduction, it suffices to show that

$$(V_1 V^l, \langle A''_o \prec_{\mathcal{V}} A'_o \rangle \langle \sigma''_A \prec_{\mathcal{V}} \sigma'_A \rangle (V_2 \langle A'_i \prec_{\mathcal{V}} A''_i \rangle V^r)) \in \mathcal{E}_{k'}^\sim \llbracket c_\sigma \circ e_\sigma \rrbracket \mathcal{V}^\sim \llbracket c_o \circ e_o \rrbracket.$$

By the induction hypothesis applied twice, it suffices to show

$$(V_1 V^l, (V_2 \langle A'_i \prec_{\mathcal{V}} A''_i \rangle V^r)) \in \mathcal{E}_{k'}^\sim \llbracket c_\sigma \rrbracket \mathcal{V}^\sim \llbracket c_o \rrbracket.$$

Finally, it suffices by the soundness of the term precision congruence rule for function application (Lemma D.25 to show that $(V_1, V_2) \in \mathcal{V}_{k'}^\sim \llbracket c_i \rightarrow_{c_\sigma} c_o \rrbracket$, and that

$$(V^l, \langle A'_i \prec_{\mathcal{V}} A''_i \rangle V^r) \in \mathcal{E}_{k'}^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket.$$

The former is true by our assumption on V_1 and V_2 . The latter follows by the induction hypothesis and our assumption on V^l and V^r .

□

LEMMA D.51 (VALUPL-GENERAL).

$$\frac{\begin{array}{l} \Sigma \mid \Gamma^\square \vdash_{d_\sigma} c : A \sqsubseteq A' \\ \Sigma \mid \Gamma^\square \vdash_{d_\sigma} e : A' \sqsubseteq A'' \\ \Sigma \mid \Gamma^\square \vdash_{d_\sigma} M \sqsubseteq N : c \circ e \end{array}}{\Sigma \mid \Gamma^\square \vdash_{d_\sigma} \langle A' \prec_{\mathcal{V}} A \rangle M \sqsubseteq N : e}$$

PROOF. Let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^\sim \llbracket \Gamma^\square \rrbracket$. We need to show that

$$(\langle A' \prec_{\mathcal{V}} A \rangle M[\gamma_1], N[\gamma_2]) \in \mathcal{E}_j^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket e \rrbracket.$$

By monadic bind (Lemma D.18), with $E_1 = \langle A' \prec_{\mathcal{V}} A \rangle \bullet$ and $E_2 = \bullet$, it suffices to show

$$((A' \prec_{\gamma} A) V_1, V_2) \in \mathcal{E}_j^{\sim} \llbracket d_{\sigma} \rrbracket \mathcal{V}^{\sim} \llbracket e \rrbracket,$$

where $k \leq j$ and $(V_1, V_2) \in \mathcal{V}_k^{\sim} \llbracket c \circ e \rrbracket$.

We continue by cases on c . The case $c = \text{bool}$ is similar to that in the previous lemma, so we skip to considering the case $c = c_i \rightarrow_{c_{\sigma}} c_o$. By inversion, we see that $e = e_i \rightarrow_{e_{\sigma}} e_o$.

We have $A = A_i \rightarrow_{\delta} A_o$ and $A' = A'_i \rightarrow_{\delta'} A'_o$, and also Thus, we have that $c \circ e = (c_i \circ e_i) \rightarrow_{c_{\sigma} \circ e_{\sigma}} (c_o \circ e_o)$.

We need to show that

$$((A'_i \rightarrow_{\delta'} A'_o) \prec_{\gamma} (A_i \rightarrow_{\delta} A_o)) M[\gamma_1], N[\gamma_2] \in \mathcal{E}_j^{\sim} \llbracket d_{\sigma} \rrbracket \mathcal{V}^{\sim} \llbracket e_i \rightarrow_{e_{\sigma}} e_o \rrbracket.$$

Similar to before, it suffices to show that these terms are related at $\mathcal{V}_k^{\sim} \llbracket e_i \rightarrow_{e_{\sigma}} e_o \rrbracket$. This is similar to proof of the previous lemma, and hence omitted. \square

LEMMA D.52 (VALDNL-GENERAL).

$$\frac{\begin{array}{l} \Sigma \mid \Gamma^{\sqsubseteq} \vdash_{d_{\sigma}} c : A \sqsubseteq A' \\ \Sigma \mid \Gamma^{\sqsubseteq} \vdash_{d_{\sigma}} e : A' \sqsubseteq A'' \\ \Sigma \mid \Gamma^{\sqsubseteq} \vdash_{d_{\sigma}} M \sqsubseteq N : e \end{array}}{\Sigma \mid \Gamma^{\sqsubseteq} \vdash_{d_{\sigma}} \langle A \leftarrow A' \rangle M \sqsubseteq N : c \circ e}$$

PROOF. This proof is dual to the proof of ValUpR-general (Lemma D.50) and is hence omitted. \square

LEMMA D.53 (VALDNR-GENERAL).

$$\frac{\begin{array}{l} \Sigma \mid \Gamma^{\sqsubseteq} \vdash_{d_{\sigma}} c : A \sqsubseteq A' \\ \Sigma \mid \Gamma^{\sqsubseteq} \vdash_{d_{\sigma}} e : A' \sqsubseteq A'' \\ \Sigma \mid \Gamma^{\sqsubseteq} \vdash_{d_{\sigma}} M \sqsubseteq N : c \circ e \end{array}}{\Sigma \mid \Gamma^{\sqsubseteq} \vdash_{d_{\sigma}} M \sqsubseteq \langle A' \leftarrow A'' \rangle N : c}$$

PROOF. This proof is dual to the proof of ValUpL-general (Lemma D.51) and is hence omitted. \square

LEMMA D.54 (EFFUPR-GENERAL).

$$\frac{\begin{array}{l} d_{\sigma} : \sigma \sqsubseteq \sigma' \\ d'_{\sigma} : \sigma' \sqsubseteq \sigma'' \\ \Sigma \mid \Gamma^{\sqsubseteq} \vdash_{d_{\sigma}} M \sqsubseteq N : c \end{array}}{\Sigma \mid \Gamma^{\sqsubseteq} \vdash_{d_{\sigma} \circ d'_{\sigma}} M \sqsubseteq \langle \sigma'' \prec_{\gamma} \sigma' \rangle N : c}$$

PROOF. Let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim} \llbracket \Gamma^{\sqsubseteq} \rrbracket$. We need to show that

$$(M, \langle \sigma'' \prec_{\gamma} \sigma' \rangle N) \in \mathcal{E}_j^{\sim} \llbracket d_{\sigma} \circ d'_{\sigma} \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket.$$

We prove this statement by Löb induction (Lemma D.16). That is, assume for all $k \leq j$ and all $(M', N') \in (\blacktriangleright \mathcal{E}^{\sim} \llbracket d_{\sigma} \rrbracket)_k (\mathcal{V}^{\sim} \llbracket c \rrbracket)$, we have

$$(M', \langle \sigma'' \prec_{\gamma} \sigma' \rangle N') \in (\blacktriangleright \mathcal{E}^{\sim} \llbracket d_{\sigma} \circ d'_{\sigma} \rrbracket)_k (\mathcal{V}^{\sim} \llbracket c \rrbracket).$$

Let $(M, N) \in \mathcal{E}_j^{\sim} \llbracket d_{\sigma} \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket$. We need to show

$$(M, \langle \sigma'' \prec_{\gamma} \sigma' \rangle N) \in \mathcal{E}_j^{\sim} \llbracket d_{\sigma} \circ d'_{\sigma} \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket.$$

We proceed by cases on d'_{σ} . The case $d'_{\sigma} = ?$ is immediate, so consider $d'_{\sigma} = \text{inj}(d_c)$, where $d_c : \sigma_c \sqsubseteq \Sigma \mid_{\text{supp}(\sigma_c)}$. In this case, we know that $\sigma'' = ?$. Furthermore, we have

$$d_\sigma \circ d'_\sigma = d_\sigma \circ (\text{inj}(d_c)) = \text{inj}(d_\sigma \circ d_c).$$

Thus, we need to show

$$(M, \langle ? \frown \sigma' \rangle N) \in \mathcal{E}_j^{\sim}[\text{inj}(d_\sigma \circ d_c)]\mathcal{V}^{\sim}[c].$$

By monadic bind (Lemma D.18), it will suffice to consider the following cases:

- Let $k \leq j$ and let $(V_1, V_2) \in \mathcal{V}_k^{\sim}[c]$. We need to show

$$(V_1, \langle ? \frown \sigma' \rangle V_2) \in \mathcal{E}_k^{\sim}[\text{inj}(d_\sigma \circ d_c)]\mathcal{V}^{\sim}[c].$$

By anti-reduction, it suffices to show that

$$(V_1, V_2) \in \mathcal{E}_k^{\sim}[\text{inj}(d_\sigma \circ d_c)]\mathcal{V}^{\sim}[c].$$

As V_1 and V_2 are values, it suffices by Lemma D.6 to show that $(V_1, V_2) \in \mathcal{V}_k^{\sim}[c]$, which is true by assumption.

- Let $k \leq j$ and $\varepsilon @ c_\varepsilon \rightsquigarrow d_\varepsilon \in d_\sigma$ be an effect that is caught by $\langle ? \frown \sigma' \rangle$. Let $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^{\sim}[c_\varepsilon])_k$, and let $E^l \# \varepsilon$ and $E^r \# \varepsilon$ be evaluation contexts such that $(x^l.E^l[x^l], x^r.E^r[x^r]) \in (\blacktriangleright \mathcal{K}^{\sim}[d_\varepsilon])_k(\mathcal{E}^{\sim}[d_\sigma])\mathcal{V}^{\sim}[c]$. We need to show that

$$\begin{aligned} & (E^l[\text{raise } \varepsilon @ c_\varepsilon^l \rightsquigarrow d_\varepsilon^l(V^l)], \\ & \langle ? \frown \sigma' \rangle E^r[\text{raise } \varepsilon @ c_\varepsilon^r \rightsquigarrow d_\varepsilon^r(V^r)]) \in \mathcal{E}_k^{\sim}[\text{inj}(d_\sigma \circ d_c)]\mathcal{V}^{\sim}[c]. \end{aligned}$$

By anti-reduction, it suffices to show that

$$\begin{aligned} & (E^l[\text{raise } \varepsilon @ c_\varepsilon^l \rightsquigarrow d_\varepsilon^l(V^l)], \\ & \text{let } y = \langle d_\varepsilon^r \llcorner d_\varepsilon^r \rangle \text{raise } \varepsilon @ c_\varepsilon^2 \rightsquigarrow d_\varepsilon^2(\langle c_\varepsilon^2 \frown c_\varepsilon^r \rangle V^r) \text{ in } \langle ? \frown \sigma' \rangle E^r[y]) \\ & \in \mathcal{E}_k^{\sim}[\text{inj}(d_\sigma \circ d_c)]\mathcal{V}^{\sim}[c]. \end{aligned}$$

Let V'' be the term to which $\langle c_\varepsilon^2 \frown c_\varepsilon^r \rangle V^r$ steps. By anti-reduction, it suffices to show

$$\begin{aligned} & (E^l[\text{raise } \varepsilon @ c_\varepsilon^l \rightsquigarrow d_\varepsilon^l(V^l)], \\ & \text{let } y = \langle d_\varepsilon^r \llcorner d_\varepsilon^r \rangle \text{raise } \varepsilon(V'') \text{ in } \langle ? \frown \sigma' \rangle E^r[y]) \\ & \in \mathcal{E}_k^{\sim}[\text{inj}(d_\sigma \circ d_c)]\mathcal{V}^{\sim}[c]. \end{aligned}$$

As neither term steps, it suffices to show they are related in $\mathcal{R}_k^{\sim}[\text{inj}(d_\sigma \circ d_c)]\mathcal{V}^{\sim}[c]$. To this end, we need to show (1) $(V^l, V'') \in (\blacktriangleright \mathcal{V}^{\sim}[c_\varepsilon \circ c_\varepsilon^r])_k$, and (2) given $k' \leq k$ and $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^{\sim}[d_\varepsilon \circ d_\varepsilon^r])_{k'}$, we have

$$\begin{aligned} & (E^l[V_1], \\ & \text{let } y = \langle d_\varepsilon^r \llcorner d_\varepsilon^r \rangle V_2 \text{ in } \langle ? \frown \sigma' \rangle E^r[y]) \\ & \in (\blacktriangleright \mathcal{E}^{\sim}[\text{Inj}(I, d_\sigma \circ d_c)])_{k'}(\mathcal{V}^{\sim}[c]). \end{aligned}$$

To show (1), it suffices by forward reduction to show that $(V^l, \langle c_\varepsilon^2 \frown c_\varepsilon^r \rangle V^r) \in (\blacktriangleright \mathcal{V}^{\sim}[c_\varepsilon \circ c_\varepsilon^r])_k$. This follows inductively from ValUpR (which we are proving simultaneously and can therefore apply at smaller types), and our assumption on V^l and V^r .

To show (2), let V'_2 be the value to which $\langle d'_\varepsilon \ll d_\varepsilon^? \rangle V_2$ steps. It suffices by anti-reduction to show

$$(E^l[V_1], \langle ? \prec_\varepsilon \sigma' \rangle E^r[V'_2]) \in (\blacktriangleright \mathcal{E}^\sim \llbracket \text{Inj}(I, d_\sigma \circ d_c) \rrbracket)_{k'}(\mathcal{V}^\sim \llbracket c \rrbracket).$$

By the Löb induction hypothesis, it suffices to show that

$$(E^l[V_1], E^r[V'_2]) \in (\blacktriangleright \mathcal{E}^\sim \llbracket d_\sigma \rrbracket)_{k'}(\mathcal{V}^\sim \llbracket c \rrbracket).$$

By our assumption on E^l and E^r , it suffices to show that $(V_1, V'_2) \in (\blacktriangleright \mathcal{V}^\sim \llbracket d_\varepsilon \rrbracket)_{k'}$. By forward reduction, it suffices to show that

$$(V_1, \langle d'_\varepsilon \ll d_\varepsilon^? \rangle V_2) \in (\blacktriangleright \mathcal{E}^\sim \llbracket d_\sigma \rrbracket)_{k'}(\mathcal{V}^\sim \llbracket d_\varepsilon \rrbracket).$$

Now inductively by ValDnR, it suffices to show $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^\sim \llbracket d_\varepsilon \circ d'_\varepsilon \rrbracket)_{k'}$, which is our assumption.

The case where d'_σ is a concrete effect precision derivation is similar to the above. \square

LEMMA D.55 (EFFUPL-GENERAL).

$$\frac{\begin{array}{l} d_\sigma : \sigma \sqsubseteq \sigma' \\ d'_\sigma : \sigma' \sqsubseteq \sigma'' \\ \Sigma \mid \Gamma^\sqsubseteq \vdash_{d_\sigma \circ d'_\sigma} M \sqsubseteq N : c \end{array}}{\Sigma \mid \Gamma^\sqsubseteq \vdash_{d'_\sigma} \langle \sigma' \prec_\varepsilon \sigma \rangle M \sqsubseteq N : c}$$

PROOF. This is proved similarly to the above. \square

LEMMA D.56 (EFFDNL-GENERAL).

$$\frac{\begin{array}{l} d_\sigma : \sigma \sqsubseteq \sigma' \\ d'_\sigma : \sigma' \sqsubseteq \sigma'' \\ \Sigma \mid \Gamma^\sqsubseteq \vdash_{d'_\sigma} M \sqsubseteq N : c \end{array}}{\Sigma \mid \Gamma^\sqsubseteq \vdash_{d_\sigma \circ d'_\sigma} \langle \sigma \ll \sigma' \rangle M \sqsubseteq N : c}$$

PROOF. We prove this by Löb induction (Lemma D.16). That is, assume for all $k \leq j$ and all $(M', N') \in (\blacktriangleright \mathcal{E}^\sim \llbracket d'_\sigma \rrbracket)_k(\mathcal{V}^\sim \llbracket c \rrbracket)$, we have

$$(\langle \sigma \ll \sigma' \rangle M', N') \in (\blacktriangleright \mathcal{E}^\sim \llbracket d_\sigma \circ d'_\sigma \rrbracket)_k(\mathcal{V}^\sim \llbracket c \rrbracket).$$

Let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^\sim \llbracket \Gamma^\sqsubseteq \rrbracket$, and let $(M, N) \in \mathcal{E}_j^\sim \llbracket d'_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket$. We need to show

$$(\langle \sigma \ll \sigma' \rangle M, N) \in \mathcal{E}_j^\sim \llbracket d_\sigma \circ d'_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket.$$

By monadic bind (Lemma D.18) and the fact that effect casts are the identity on values, it will suffice to show the following:

Let $k \leq j$ and $\varepsilon @_{c_\varepsilon} \rightsquigarrow d_\varepsilon \in d'_\sigma$ be an effect that is caught by $\langle \sigma \ll \sigma' \rangle \bullet$. Let $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^\sim \llbracket c_\varepsilon \rrbracket)_k$, and let $E^l \# \varepsilon$ and $E^r \# \varepsilon$ be evaluation contexts such that $(x^l.E^l[x^l], x^r.E^r[x^r]) \in (\blacktriangleright \mathcal{K}^\sim \llbracket d_\varepsilon \rrbracket)_k(\mathcal{E}^\sim \llbracket d'_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket)$. We need to show that

$$\begin{aligned}
& (\langle \sigma \leftarrow \sigma' \rangle E^l [\text{raise } \varepsilon @ c_\varepsilon^l \rightsquigarrow d_\varepsilon^l (V^l)], \\
& E^r [\text{raise } \varepsilon @ c_\varepsilon^r \rightsquigarrow d_\varepsilon^r (V^r)] N) \\
& \in \mathcal{E}_j^\sim \llbracket d_\sigma \circ d'_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket.
\end{aligned}$$

Note that if $\varepsilon \notin \sigma$, then the left hand side steps to \mathcal{U} , in which case we are finished by ErrBot (Lemma D.47). Otherwise, the proof proceeds analogously to EffUpR (Lemma D.54), with upcasts and downcasts interchanged. \square

LEMMA D.57 (EFFDNR-GENERAL).

$$\frac{
\begin{array}{l}
d_\sigma : \sigma \sqsubseteq \sigma' \\
d'_\sigma : \sigma' \sqsubseteq \sigma'' \\
\Sigma \mid \Gamma^\square \vdash_{d_\sigma \circ d'_\sigma} M \sqsubseteq N : c
\end{array}
}{
\Sigma \mid \Gamma^\square \vdash_{d_\sigma} M \sqsubseteq \langle \sigma' \leftarrow \sigma'' \rangle N : c
}$$

PROOF. We prove this statement by Löb induction (Lemma D.16). That is, assume for all $k \leq j$ and all $(M', N') \in (\blacktriangleright \mathcal{E}^\sim \llbracket d_\sigma \circ d'_\sigma \rrbracket)_k (\mathcal{V}^\sim \llbracket c \rrbracket)$, we have

$$(M', \langle \sigma' \leftarrow \sigma'' \rangle N') \in (\blacktriangleright \mathcal{E}^\sim \llbracket d_\sigma \rrbracket)_k (\mathcal{V}^\sim \llbracket c \rrbracket).$$

Let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^\sim \llbracket \Gamma^\square \rrbracket$, and let $(M, N) \in \mathcal{E}_j^\sim \llbracket d_\sigma \circ d'_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket$. We need to show

$$(M, \langle \sigma' \leftarrow \sigma'' \rangle N) \in \mathcal{E}_j^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket.$$

By monadic bind (Lemma D.18) and the fact that effect casts are the identity on values, it will suffice to show the following:

Let $k \leq j$ and $\varepsilon @ c_\varepsilon \rightsquigarrow d_\varepsilon \in d_\sigma \circ d'_\sigma$ be an effect that is caught by $\langle \sigma' \leftarrow \sigma'' \rangle \bullet$. Let $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^\sim \llbracket c_\varepsilon \rrbracket)_k$, and let $E^l \# \varepsilon$ and $E^r \# \varepsilon$ be evaluation contexts such that $(x^l.E^l[x^l], x^r.E^r[x^r]) \in (\blacktriangleright \mathcal{K}^\sim \llbracket d_\varepsilon \rrbracket)_k (\mathcal{E}^\sim \llbracket d_\sigma \circ d'_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket)$. We need to show that

$$\begin{aligned}
& (E^l [\text{raise } \varepsilon @ c_\varepsilon^l \rightsquigarrow d_\varepsilon^l (V^l)], \\
& \langle \sigma' \leftarrow \sigma'' \rangle E^r [\text{raise } \varepsilon @ c_\varepsilon^r \rightsquigarrow d_\varepsilon^r (V^r)]) \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket.
\end{aligned}$$

First note that by Lemma D.21, there exist c_1, c_2, d_1 , and d_2 such that $c_\varepsilon = c_1 \circ c_2$ and $d_\varepsilon = d_1 \circ d_2$ and $\varepsilon @ c_1 \rightsquigarrow d_1 \in d_\sigma$ and $\varepsilon @ c_2 \rightsquigarrow d_2 \in d'_\sigma$. In particular, this that $\varepsilon \in \sigma'$, so the downcast from σ'' to σ' does not fail. Let $c^L = c_1^l (= c_\varepsilon^l)$, $c^M = c_1^r = c_2^l$, and $c^R = c_2^r (= c_\varepsilon^r)$, and likewise define d^L, d^M and d^R .

By anti-reduction, it suffices to show that

$$\begin{aligned}
& (E^l [\text{raise } \varepsilon @ c^L \rightsquigarrow d^L (V^l)], \\
& \text{let } y = \langle d^R \leftarrow d^M \rangle \text{raise } \varepsilon @ c^M \rightsquigarrow d^M (\langle c^M \leftarrow c^R \rangle V^r) \text{ in } \langle \sigma' \leftarrow \sigma'' \rangle E^r [y]) \\
& \in \mathcal{E}_k^\sim \llbracket d_\sigma \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket.
\end{aligned}$$

Let V^{rr} be the term to which $\langle c^M \leftarrow c^R \rangle V^r$ steps. By anti-reduction, it suffices to show

$$\begin{aligned}
& (E^l[\text{raise } \varepsilon@c^L \rightsquigarrow D^L(V^l)], \\
& \text{let } y = \langle d^R \leftarrow d^M \rangle \text{raise } \varepsilon@c^M \rightsquigarrow d^M(V^{rr}) \text{ in } \langle \sigma' \leftarrow \sigma'' \rangle E^r[y]) \\
& \in \mathcal{E}_k^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket.
\end{aligned}$$

As neither term steps, it suffices to show they are related in $\mathcal{R}_k^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket$. To this end, we need to show (1) $(V^l, V^{rr}) \in (\blacktriangleright \mathcal{V}^{\sim} \llbracket c_1 \rrbracket)_k$, and (2) given $k' \leq k$ and $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^{\sim} \llbracket d_1 \rrbracket)_{k'}$, we have

$$\begin{aligned}
& (E^l[V_1], \\
& \text{let } y = \langle d^R \leftarrow d^M \rangle V_2 \text{ in } \langle \sigma' \leftarrow \sigma'' \rangle E^r[y]) \\
& \in (\blacktriangleright \mathcal{E}^{\sim} \llbracket d_\sigma \rrbracket)_{k'} (\mathcal{V}^{\sim} \llbracket c \rrbracket).
\end{aligned}$$

(1) follows from forward reduction and the inductive hypothesis for value types. To show (2), let V'_2 be the value to which $\langle d^R \leftarrow d^M \rangle V_2$ steps. It suffices by anti-reduction to show

$$\begin{aligned}
& (E^l[V_1], \langle \sigma'' \leftarrow \sigma' \rangle E^r[V'_2]) \\
& \in (\blacktriangleright \mathcal{E}^{\sim} \llbracket d_\sigma \rrbracket)_{k'} (\mathcal{V}^{\sim} \llbracket c \rrbracket).
\end{aligned}$$

By the Löb induction hypothesis, it suffices to show that

$$\begin{aligned}
& (E^l[V_1], E^r[V'_2]) \\
& \in (\blacktriangleright \mathcal{E}^{\sim} \llbracket d_\sigma \circ d'_\sigma \rrbracket)_{k'} (\mathcal{V}^{\sim} \llbracket c \rrbracket).
\end{aligned}$$

By our assumption on E^l and E^r , it suffices to show that $(V_1, V'_2) \in (\blacktriangleright \mathcal{V}^{\sim} \llbracket d_\varepsilon \rrbracket)_{k'}$. By forward reduction, it suffices to show that

$$(V_1, \langle d^R \leftarrow d^M \rangle V_2) \in (\blacktriangleright \mathcal{E}^{\sim} \llbracket d_\sigma \rrbracket)_{k'} (\mathcal{V}^{\sim} \llbracket d_\varepsilon \rrbracket).$$

Now inductively by ValUpR, it suffices to show $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^{\sim} \llbracket d_1 \rrbracket)_{k'}$, which is our assumption.

The case where d'_σ is a concrete effect precision derivation is similar to the above. \square

LEMMA D.58 (VALUP-EVAL).

$$\langle B \leftarrow A \rangle M \equiv \text{let } x = M \text{ in } \langle B \leftarrow A \rangle x$$

PROOF. We show one direction of the equivalence; the other is symmetric. Let j be arbitrary and let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim} \llbracket \Gamma \rrbracket$. We need to show

$$(((\langle B \leftarrow A \rangle M)[\gamma_1], (\text{let } x = M \text{ in } \langle B \leftarrow A \rangle x)[\gamma_2]) \in \mathcal{E}_j^{\sim} \llbracket \sigma \rrbracket \mathcal{V}^{\sim} \llbracket B \rrbracket).$$

By Monadic Bind (Lemma D.18) and reflexivity, it will suffice to show that for all $k \leq j$ let $(V_1, V_2) \in \mathcal{V}_k^{\sim} \llbracket A \rrbracket$, we have

$$(((\langle B \leftarrow A \rangle V_1), (\text{let } x = V_2 \text{ in } \langle B \leftarrow A \rangle x)) \in \mathcal{E}_k^{\sim} \llbracket \sigma \rrbracket \mathcal{V}^{\sim} \llbracket B \rrbracket).$$

By anti-reduction, it suffices to show

$$(((\langle B \leftarrow A \rangle V_1), (\langle B \leftarrow A \rangle V_2)) \in \mathcal{E}_k^{\sim} \llbracket \sigma \rrbracket \mathcal{V}^{\sim} \llbracket B \rrbracket).$$

By congruence, it suffices to show

$$(V_1, V_2) \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![B]\!].$$

This follows from our assumption on V_1 and V_2 . □

LEMMA D.59 (VALDNEVAL).

$$\langle A \leftarrow B \rangle M \equiv \text{let } x = M \text{ in } \langle A \leftarrow B \rangle x$$

PROOF. Dual to the above. □

LEMMA D.60 (CAST-RETRACTION). *let $A \sqsubseteq B$ and $\sigma \sqsubseteq \sigma'$, and let $c : A \sqsubseteq B$ and $d_\sigma : \sigma \sqsubseteq \sigma'$. Let $\Sigma \mid \Gamma^{\sqsubseteq} \vdash_\sigma M \sqsubseteq N : A$. The following hold:*

- (1) $\Sigma \mid \Gamma^{\sqsubseteq} \vDash_\sigma \langle A \leftarrow B \rangle \langle B \leftarrow A \rangle M \sqsubseteq N : A$
- (2) $\Sigma \mid \Gamma^{\sqsubseteq} \vDash_\sigma \langle \sigma \leftarrow \sigma' \rangle \langle \sigma' \leftarrow \sigma \rangle M \sqsubseteq N : A$

PROOF. We prove stronger, “pointwise” version of the above statements. Namely, we assume $(M, N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!]$, and show, for example, that $(\langle A \leftarrow B \rangle \langle B \leftarrow A \rangle M, N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!]$. The proof is by simultaneous induction on the derivations c and d_σ .

- (1) Let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim}[\![A]\!]$. Suppose $(M, N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!]$. We need to show

$$(\langle A \leftarrow B \rangle \langle B \leftarrow A \rangle M, N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!].$$

By monadic bind (Lemma D.18), it suffices to show that

$$(\langle A \leftarrow B \rangle \langle B \leftarrow A \rangle V_1, V_2) \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!],$$

where $k \leq j$ and $(V_1, V_2) \in \mathcal{V}_k^{\sim}[\![A]\!]$.

We proceed by induction on the precision derivation c . If $c = \text{bool}$, then we need to show

$$(\langle \text{bool} \leftarrow \text{bool} \rangle \langle \text{bool} \leftarrow \text{bool} \rangle V_1, V_2) \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![\text{bool}]\!].$$

According to the operational semantics, we have that

$$\langle \text{bool} \leftarrow \text{bool} \rangle \langle \text{bool} \leftarrow \text{bool} \rangle V_1 \mapsto^2 V_1.$$

So by anti-reduction (Lemma D.8), it suffices to show that $(V_1, V_2) \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![\text{bool}]\!]$, which follows from our assumption.

If $c = c_i \rightarrow_{c_\sigma} c_o$, then $A = A_i \rightarrow_{\sigma_A} A_o$ and $B = B_i \rightarrow_{\sigma_B} B_o$. We need to show

$$\begin{aligned} & ((\langle A_i \rightarrow_{\sigma_A} A_o \rangle \leftarrow \langle B_i \rightarrow_{\sigma_B} B_o \rangle) \langle \langle B_i \rightarrow_{\sigma_B} B_o \rangle \leftarrow \langle A_i \rightarrow_{\sigma_A} A_o \rangle \rangle V_1, V_2) \\ & \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A_i \rightarrow_{\sigma_A} A_o]\!]. \end{aligned}$$

As both of these are values, it suffices to show that they are related in $\mathcal{V}^{\sim}[\![A_i \rightarrow_{\sigma_A} A_o]\!]$. To this end, let $k' \leq k$ and let $(V^l, V^r) \in \mathcal{V}_{k'}^{\sim}[\![A_i]\!]$. We need to show that

$$\begin{aligned} & (((\langle A_i \rightarrow_{\sigma_A} A_o \rangle \leftarrow \langle B_i \rightarrow_{\sigma_B} B_o \rangle) \langle \langle B_i \rightarrow_{\sigma_B} B_o \rangle \leftarrow \langle A_i \rightarrow_{\sigma_A} A_o \rangle \rangle V_1)^l, \\ & V_2^r) \\ & \in \mathcal{E}_{k'}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_o]\!]. \end{aligned}$$

The former term steps, so by anti-reduction, it suffices to show that

$$\begin{aligned} & \langle A_o \Leftarrow B_o \rangle & \langle \sigma_A \Leftarrow \sigma_B \rangle & (\langle \langle \langle B_i \rightarrow_{\sigma_B} B_o \rangle \Leftarrow (A_i \rightarrow_{\sigma_A} A_o) \rangle V_1 \rangle \langle B_i \Leftarrow A_i \rangle V^l), V_2 V^r) \\ & \in \mathcal{E}_{k'}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_o]\!]. \end{aligned}$$

Let V'^l be the value to which $\langle B_i \Leftarrow A_i \rangle V^l$ steps. By anti-reduction, it suffices to show that

$$\begin{aligned} & \langle \langle A_o \Leftarrow B_o \rangle \langle \sigma_A \Leftarrow \sigma_B \rangle \\ & \quad \langle \langle B_o \Leftarrow A_o \rangle \langle \sigma_B \Leftarrow \sigma_A \rangle \\ & \quad \quad (V_1 \langle A_i \Leftarrow B_i \rangle V'^l), \\ & \quad V_2 V^r) \\ & \in \mathcal{E}_{k'}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_o]\!]. \end{aligned}$$

We will appeal to transitivity (Lemma D.66). We continue by cases on \sim . First assume \sim is $<$. Let V'' be the value to which $\langle B_i \Leftarrow A_i \rangle V^r$ steps. If we show (1)

$$\begin{aligned} & \langle \langle A_o \Leftarrow B_o \rangle \langle \sigma_A \Leftarrow \sigma_B \rangle \\ & \quad \langle \langle B_o \Leftarrow A_o \rangle \langle \sigma_B \Leftarrow \sigma_A \rangle \\ & \quad \quad (V_1 \langle A_i \Leftarrow B_i \rangle V'^l), \\ & \quad \langle A_o \Leftarrow B_o \rangle \langle B_o \Leftarrow A_o \rangle \\ & \quad \langle \langle \sigma_A \Leftarrow \sigma_B \rangle \langle \sigma_B \Leftarrow \sigma_A \rangle \\ & \quad \quad (V_2 \langle A_i \Leftarrow B_i \rangle V'') \rangle) \\ & \in \mathcal{E}_{k'}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_o]\!]. \end{aligned}$$

and (2)

$$\begin{aligned} & \langle \langle A_o \Leftarrow B_o \rangle \langle B_o \Leftarrow A_o \rangle \\ & \quad \langle \langle \sigma_A \Leftarrow \sigma_B \rangle \langle \sigma_B \Leftarrow \sigma_A \rangle \\ & \quad \quad (V_2 \langle A_i \Leftarrow B_i \rangle V'') \rangle, \\ & \quad V_2 V^r) \\ & \in \mathcal{E}_{\omega}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_o]\!], \end{aligned}$$

then we will be finished by transitivity.

To show (1), first note that by monotonicity of casts (Lemma D.65), it suffices to show that

$$\begin{aligned} & \langle \langle \sigma_A \Leftarrow \sigma_B \rangle \\ & \quad \langle \langle B_o \Leftarrow A_o \rangle \langle \sigma_B \Leftarrow \sigma_A \rangle \\ & \quad \quad (V_1 \langle A_i \Leftarrow B_i \rangle V'^l), \\ & \quad \langle B_o \Leftarrow A_o \rangle \\ & \quad \langle \langle \sigma_A \Leftarrow \sigma_B \rangle \langle \sigma_B \Leftarrow \sigma_A \rangle \\ & \quad \quad (V_2 \langle A_i \Leftarrow B_i \rangle V'') \rangle) \\ & \in \mathcal{E}_{k'}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![B_o]\!]. \end{aligned}$$

Then by commutativity of casts (Corollary D.63), it suffices to show

$$\begin{aligned} & (\langle \sigma_B \prec \sigma_A \rangle (V_1 \langle A_i \leftarrow B_i \rangle V^l), \\ & \langle \sigma_B \prec \sigma_A \rangle (V_2 \langle A_i \leftarrow B_i \rangle V^r)) \\ & \in \mathcal{E}_{k'}^{\sim}[\![\sigma_B]\!] \mathcal{V}^{\sim}[\![A_o]\!]. \end{aligned}$$

By monotonicity of casts again, it suffices to show

$$\begin{aligned} & ((V_1 \langle A_i \leftarrow B_i \rangle V^l), \\ & (V_2 \langle A_i \leftarrow B_i \rangle V^r)) \\ & \in \mathcal{E}_{k'}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_o]\!]. \end{aligned}$$

By soundness of the precision rule for function application, it suffices to show that $(V_1, V_2) \in \mathcal{V}_k^{\sim}[\![A_i \rightarrow_{\sigma_A} A_o]\!]$ and that $(\langle A_i \leftarrow B_i \rangle V^l, \langle A_i \leftarrow B_i \rangle V^r) \in \mathcal{E}_k^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_i]\!]$. The former holds by assumption, and to show the latter, it suffices by forward reduction to show $(\langle A_i \leftarrow B_i \rangle \langle B_i \prec A_i \rangle V^l, \langle A_i \leftarrow B_i \rangle \langle B_i \prec A_i \rangle V^r) \in \mathcal{E}_k^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_i]\!]$. This follows from the inductive hypothesis and assumption on V^l and V^r .

To show (2), it suffices by the inductive hypothesis applied twice to show

$$\begin{aligned} & ((V_2 \langle A_i \leftarrow B_i \rangle V^r), V_2 V^r) \\ & \in \mathcal{E}_{\omega}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_o]\!], \end{aligned}$$

By forward reduction, it suffices to show

$$\begin{aligned} & ((V_2 \langle A_i \leftarrow B_i \rangle V^r), V_2 V^r) \\ & \in \mathcal{E}_{\omega}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_o]\!], \end{aligned}$$

By soundness of function application, it suffices to show that V_2 is related to itself at $\mathcal{V}_{\omega}^{\sim}[\![A_i \rightarrow_{\sigma_A} A_o]\!]$ and that $(\langle A_i \leftarrow B_i \rangle V^r, V^r) \in \mathcal{E}_{\omega}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_i]\!]$. The former holds by reflexivity (Corollary D.30), and to show the latter it suffices by forward reduction to show that

$$(\langle A_i \leftarrow B_i \rangle \langle B_i \prec A_i \rangle V^r, V^r) \in \mathcal{E}_{\omega}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_i]\!],$$

which follows by the inductive hypothesis and reflexivity.

The case when \sim is $<$ is analogous.

- (2) Let $(\gamma_1, \gamma_2) \in \mathcal{G}_j^{\sim}[\![A]\!]$. We use Löb induction. We assume that for all $k \leq j$ and all related terms $(M', N') \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma]\!])_k(\mathcal{V}^{\sim}[\![A]\!])$, we have

$$(\langle \sigma \leftarrow \sigma' \rangle \langle \sigma' \prec \sigma \rangle M', N') \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma]\!])_k(\mathcal{V}^{\sim}[\![A]\!]).$$

Let $(M, N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!]$. We need to show that

$$(\langle \sigma \leftarrow \sigma' \rangle \langle \sigma' \prec \sigma \rangle M, N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!].$$

By monadic bind (Lemma D.18), it suffices to consider the following cases:

(a) Let $k \leq j$ and $(V_1, V_2) \in \mathcal{V}_k^\sim[[A]]$. We need to show

$$\langle \langle \sigma \leftarrow \sigma' \rangle \langle \sigma' \leftarrow \sigma \rangle V_1, V_2 \rangle \in \mathcal{E}_k^\sim[[\sigma]]\mathcal{V}^\sim[[A]]$$

This follows by anti-reduction and assumption.

(b) Let $k \leq j$ and let $\varepsilon @ C \rightsquigarrow D \in \sigma$. Let C' and D' be the types such that $\varepsilon @ C' \rightsquigarrow D' \in \sigma'$. Let $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^\sim[[C]])_k$ and let $E^l \# \varepsilon$ and $E^r \# \varepsilon$ be such that

$$(x^l.E^l[x^l], x^r.E^r[x^r]) \in (\blacktriangleright \mathcal{K}^\sim[[D]])_k(\mathcal{E}^\sim[[\sigma]]\mathcal{V}^\sim[[A]]).$$

We need to show that

$$\begin{aligned} & \langle \langle \sigma \leftarrow \sigma' \rangle \langle \sigma' \leftarrow \sigma \rangle E^l[\text{raise } \varepsilon(V^l)], E^r[\text{raise } \varepsilon(V^r)] \rangle \\ & \in \mathcal{E}_k^\sim[[\sigma]]\mathcal{V}^\sim[[A]]. \end{aligned}$$

The first term steps, so by anti-reduction it suffices to show

$$\begin{aligned} & \langle \langle \sigma \leftarrow \sigma' \rangle (\text{let } x = \langle D \leftarrow D' \rangle \text{raise } \varepsilon(\langle C' \leftarrow C \rangle V^l) \text{ in } \langle \sigma' \leftarrow \sigma \rangle E^l[x]), \\ & E^r[\text{raise } \varepsilon(V^r)] \rangle \\ & \in \mathcal{E}_k^\sim[[\sigma]]\mathcal{V}^\sim[[A]]. \end{aligned}$$

Let V'^l be the value to which $\langle C' \leftarrow C \rangle V^l$ steps. By anti-reduction, it suffices to show

$$\begin{aligned} & \langle \text{let } y = \langle D' \leftarrow D \rangle \text{raise } \varepsilon(\langle C \leftarrow C' \rangle V'^l) \text{ in } \langle \sigma \leftarrow \sigma' \rangle \text{let } x = \langle D \leftarrow D' \rangle y \text{ in } \langle \sigma' \leftarrow \sigma \rangle E^l[x], \\ & E^r[\text{raise } \varepsilon(V^r)] \rangle \\ & \in \mathcal{E}_k^\sim[[\sigma]]\mathcal{V}^\sim[[A]]. \end{aligned}$$

Let y' be the value to which $\langle D \leftarrow D' \rangle y$ steps. Let V''^l be the value to which $\langle C \leftarrow C' \rangle V'^l$ steps.

By anti-reduction, it suffices to show

$$\begin{aligned} & \langle \text{let } y = \langle D' \leftarrow D \rangle \text{raise } \varepsilon(V''^l) \text{ in } \langle \sigma \leftarrow \sigma' \rangle \langle \sigma' \leftarrow \sigma \rangle E^l[y'], \\ & E^r[\text{raise } \varepsilon(V^r)] \rangle \\ & \in \mathcal{E}_k^\sim[[\sigma]]\mathcal{V}^\sim[[A]]. \end{aligned}$$

Neither term steps, so it suffices to show they are related in $\mathcal{R}_k^\sim[[\sigma]]\mathcal{V}^\sim[[A]]$. To this end, we first show that $(V''^l, V^r) \in (\blacktriangleright \mathcal{V}^\sim[[C]])_k$. By forward reduction, it suffices to show that $\langle \langle C \leftarrow C' \rangle \langle C' \leftarrow C \rangle V^l, V^r \rangle \in (\blacktriangleright \mathcal{V}^\sim[[C]])_k$. This follows from the inductive hypothesis for value types and our assumption on V^l and V^r .

We now show that, given $k' \leq k$ and values $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^\sim[[D]])_{k'}$, we have

$$\begin{aligned} & \langle \text{let } y = \langle D' \leftarrow D \rangle V_1 \text{ in } \langle \sigma \leftarrow \sigma' \rangle \langle \sigma' \leftarrow \sigma \rangle E^l[y'], \\ & E^r[V_2] \rangle \\ & \in (\blacktriangleright \mathcal{E}^\sim[[\sigma]])_k(\mathcal{V}^\sim[[A]]). \end{aligned}$$

Let V'_1 be the value to which $\langle D' \leftarrow D \rangle V_1$ steps. By anti-reduction, it will suffice to show

$$\begin{aligned} & (\text{let } y = V_1' \text{ in } \langle \sigma \leftarrow \sigma' \rangle \langle \sigma' \leftarrow \sigma \rangle E^l[y'], \\ & E^r[V_2]) \\ & \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_k(\mathcal{V}^\sim[A]). \end{aligned}$$

By forward reduction, it will suffice to show

$$\begin{aligned} & (\text{let } y = V_1' \text{ in } \langle \sigma \leftarrow \sigma' \rangle \langle \sigma' \leftarrow \sigma \rangle E^l[\langle D \leftarrow D' \rangle y], \\ & E^r[V_2]) \\ & \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_k(\mathcal{V}^\sim[A]). \end{aligned}$$

By anti-reduction, it will suffice to show

$$\begin{aligned} & (\langle \sigma \leftarrow \sigma' \rangle \langle \sigma' \leftarrow \sigma \rangle E^l[\langle D \leftarrow D' \rangle V_1'], E^r[V_2]) \\ & \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_k(\mathcal{V}^\sim[A]). \end{aligned}$$

By the Löb induction hypothesis, it suffices to show that

$$\begin{aligned} & (E^l[\langle D \leftarrow D' \rangle V_1'], E^r[V_2]) \\ & \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_k(\mathcal{V}^\sim[A]). \end{aligned}$$

By forward reduction, it suffices to show

$$\begin{aligned} & (E^l[\langle D \leftarrow D' \rangle \langle D' \leftarrow D \rangle V_1], E^r[V_2]) \\ & \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_k(\mathcal{V}^\sim[A]). \end{aligned}$$

By the induction hypothesis for value types, it suffices to show

$$(E^l[V_1], E^r[V_2]) \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_k(\mathcal{V}^\sim[A]).$$

This follows by our assumption on E^l and E^r .

□

LEMMA D.61 (GRADUAL SUBTYPING). *Let $c : A \sqsubseteq B$ and $c' : A' \sqsubseteq B'$ where $A \leq A'$ and $B \leq B'$. Let $d_\sigma : \sigma_1 \sqsubseteq \sigma_2$ and $d'_\sigma : \sigma'_1 \sqsubseteq \sigma'_2$ where $\sigma_1 \leq \sigma'_1$ and $\sigma_2 \leq \sigma'_2$. Suppose $M \equiv N$. The following hold:*

(1)

$$\frac{\Sigma \mid \Gamma^\sqsubseteq \vDash_{d_\tau} M \sqsubseteq N : A}{\Sigma \mid \Gamma^\sqsubseteq \vDash_{d_\tau} \langle B \leftarrow A \rangle M \sqsubseteq \langle B' \leftarrow A' \rangle N : B'}$$

(2)

$$\frac{\Sigma \mid \Gamma^\sqsubseteq \vDash_{d_\tau} M \sqsubseteq N : B}{\Sigma \mid \Gamma^\sqsubseteq \vDash_{d_\tau} \langle A' \leftarrow B' \rangle M \sqsubseteq \langle A \leftarrow B \rangle N : A'}$$

(3)

$$\frac{\Sigma \mid \Gamma^\sqsubseteq \vDash_{\sigma_1} M \sqsubseteq N : d}{\Sigma \mid \Gamma^\sqsubseteq \vDash_{\sigma'_2} \langle \sigma_2 \leftarrow \sigma_1 \rangle M \sqsubseteq \langle \sigma'_2 \leftarrow \sigma'_1 \rangle N : d}$$

(4)

$$\frac{\Sigma \mid \Gamma^{\square} \vDash_{\sigma_2} M \sqsubseteq N : d}{\Sigma \mid \Gamma^{\square} \vDash_{\sigma'_1} \langle \sigma'_1 \leftarrow \sigma'_2 \rangle M \sqsubseteq \langle \sigma_1 \leftarrow \sigma_2 \rangle N : d}$$

PROOF. By simultaneous induction on the derivation $c' : A' \sqsubseteq B'$ and $d'_\sigma : \sigma'_1 \sqsubseteq \sigma'_2$.

(1) We need to show

$$\langle B \leftarrow A \rangle M, \langle B' \leftarrow A' \rangle N \in \mathcal{E}_j^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket B' \rrbracket.$$

By monadic bind (Lemma D.18), with $E_1 = \langle B \leftarrow A \rangle \bullet$ and $E_2 = \langle B' \leftarrow A' \rangle \bullet$, it suffices to show the following.

Let $k \leq j$ and let $(V_1, V_2) \in \mathcal{V}_k^{\sim} \llbracket A' \rrbracket$. We need to show

$$\langle B \leftarrow A \rangle V_1, \langle B' \leftarrow A' \rangle V_2 \in \mathcal{E}_k^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket B' \rrbracket.$$

We continue by cases on c' .

Case $c' = \text{bool}$. Then by inversion on the rules for subtyping of precision derivations, we have $c = \text{bool}$.

We need to show

$$\langle \text{bool} \leftarrow \text{bool} \rangle V_1, \langle \text{bool} \leftarrow \text{bool} \rangle V_2 \in \mathcal{E}_k^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket \text{bool} \rrbracket$$

This follows by anti-reduction and our assumption on V_1 and V_2 .

Case $c' = c'_i \rightarrow_{c'_\sigma} c'_o : A'_i \rightarrow_{\sigma'_A} A'_o \sqsubseteq B'_i \rightarrow_{\sigma'_B} B'_o$.

By inversion on the rules for subtyping for precision derivations, we have that $c = c_i \rightarrow_{c_\sigma} c_o$, where $c'_i \leq c_i$, and $c_\sigma \leq c'_\sigma$, and $c_o \leq c'_o$.

Our assumption then becomes $(V_1, V_2) \in \mathcal{V}_k^{\sim} \llbracket A'_i \rightarrow_{\sigma'_A} A'_o \rrbracket$. We need to show

$$\langle B_i \rightarrow_{\sigma_B} B_o \leftarrow A_i \rightarrow_{\sigma_A} A_o \rangle V_1, \langle B'_i \rightarrow_{\sigma'_B} B'_o \leftarrow A'_i \rightarrow_{\sigma'_A} A'_o \rangle V_2 \in \mathcal{E}_k^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket B'_i \rightarrow_{\sigma'_B} B'_o \rrbracket.$$

Since both terms are values, it suffices to show they are related in $\mathcal{V}_k^{\sim} \llbracket B'_i \rightarrow_{\sigma'_B} B'_o \rrbracket$. Let $k' \leq k$ and let $(V^l, V^r) \in \mathcal{V}_{k'}^{\sim} \llbracket B'_i \rrbracket$. We need to show

$$\begin{aligned} & (\langle B_i \rightarrow_{\sigma_B} B_o \leftarrow A_i \rightarrow_{\sigma_A} A_o \rangle V_1) V^l, \\ & (\langle B'_i \rightarrow_{\sigma'_B} B'_o \leftarrow A'_i \rightarrow_{\sigma'_A} A'_o \rangle V_2) V^r \\ & \in \mathcal{E}_{k'}^{\sim} \llbracket \sigma'_B \rrbracket \mathcal{V}^{\sim} \llbracket B'_o \rrbracket. \end{aligned}$$

By anti-reduction, it suffices to show

$$\begin{aligned} & \langle B_o \leftarrow A_o \rangle \langle \sigma_B \leftarrow \sigma_A \rangle (V_1 \langle A_i \leftarrow B_i \rangle V^l), \\ & \langle B'_o \leftarrow A'_o \rangle \langle \sigma'_B \leftarrow \sigma'_A \rangle (V_2 \langle A'_i \leftarrow B'_i \rangle V^r) \\ & \in \mathcal{E}_{k'}^{\sim} \llbracket \sigma'_B \rrbracket \mathcal{V}^{\sim} \llbracket B'_o \rrbracket. \end{aligned}$$

By the induction hypothesis applied twice, it suffices to show

$$\begin{aligned} & ((V_1 \langle A_i \leftarrow B_i \rangle V^l), (V_2 \langle A'_i \leftarrow B'_i \rangle V^r)) \\ & \in \mathcal{E}_{k'}^{\sim} \llbracket \sigma'_A \rrbracket \mathcal{V}^{\sim} \llbracket A'_o \rrbracket. \end{aligned}$$

By soundness of the term precision congruence rule for function application (Lemma D.25), it suffices to show that $(V_1, V_2) \in \mathcal{V}_{k'}^{\sim} \llbracket A'_i \rightarrow_{\sigma'_A} A'_o \rrbracket$, and that

$$(\langle A_i \leftarrow B_i \rangle V^l, \langle A'_i \leftarrow B'_i \rangle V^r) \in \mathcal{E}_{k'}^{\sim} \llbracket d_\sigma \rrbracket \mathcal{V}^{\sim} \llbracket A'_i \rrbracket.$$

The former holds by assumption. To show the latter, it suffices by the admissible direction of gradual subtyping rule ValDnSub (item (2) in Lemma B.1), whose proof does not depend on the present lemma, to show that $(V^l, V^r) \in \mathcal{V}_{k'}^{\sim} \llbracket B'_i \rrbracket$. This is true by assumption.

- (2) Similar to the above.
- (3) We need to show

$$(\langle \sigma_2 \prec \sigma_1 \rangle M, \langle \sigma'_2 \prec \sigma'_1 \rangle N) \in \mathcal{E}_j^{\sim} \llbracket \sigma'_2 \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket.$$

We use Löb induction. That is, we assume as our induction hypothesis that

$$(\langle \sigma_2 \prec \sigma_1 \rangle M', \langle \sigma'_2 \prec \sigma'_1 \rangle N') \in \blacktriangleright (\mathcal{E}^{\sim} \llbracket \sigma'_2 \rrbracket)_j (\mathcal{V}^{\sim} \llbracket c \rrbracket),$$

for all $(M', N') \in (\blacktriangleright \mathcal{E}^{\sim} \llbracket \sigma'_1 \rrbracket)_j (\mathcal{V}^{\sim} \llbracket c \rrbracket)$, and we show that under this assumption, we have

$$(\langle \sigma_2 \prec \sigma_1 \rangle M, \langle \sigma'_2 \prec \sigma'_1 \rangle N) \in \mathcal{E}_j^{\sim} \llbracket \sigma'_2 \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket$$

for all $(M, N) \in \mathcal{E}_j^{\sim} \llbracket \sigma'_1 \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket$.

Using Monadic Bind (Lemma D.18), we have the following cases:

- Let $k \leq j$ and $(V_1, V_2) \in \mathcal{V}_k^{\sim} \llbracket c \rrbracket$. We need to show

$$(\langle \sigma_2 \prec \sigma_1 \rangle V_1, \langle \sigma'_2 \prec \sigma'_1 \rangle V_2) \in \mathcal{E}_k^{\sim} \llbracket \sigma'_2 \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket.$$

This follows by anti-reduction and our assumption on V_1 and V_2 .

- Let $\varepsilon @ c_i \rightsquigarrow d_i \in \sigma_1$ be an effect caught by $\langle \sigma'_2 \prec \sigma'_1 \rangle$. Let $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^{\sim} \llbracket c'_i \rrbracket)_k$, and let $(E^l, E^r) \in (\blacktriangleright \mathcal{K}^{\sim} \llbracket d'_i \rrbracket)_k (\mathcal{E}^{\sim} \llbracket \sigma'_1 \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket)$. We need to show

$$\begin{aligned} & (\langle \sigma_2 \prec \sigma_1 \rangle E^l [\text{raise } \varepsilon(V^l)], \langle \sigma'_2 \prec \sigma'_1 \rangle E^r [\text{raise } \varepsilon(V^r)]) \\ & \in \mathcal{E}_k^{\sim} \llbracket \sigma'_2 \rrbracket \mathcal{V}^{\sim} \llbracket c \rrbracket. \end{aligned}$$

We continue by cases on subtyping of effect precision derivations. We show only the case d'_σ is a concrete effect precision set d'_c ; the other cases follow immediately or reduce to this one.

By inversion, we have d_σ is also a concrete effect precision set d_c where $\text{dom}(d_c) \subseteq \text{dom}(d'_c)$ and for all $\varepsilon : c \rightsquigarrow d \in d_c$, $\varepsilon : c' \rightsquigarrow d' \in d'_c$ and $c \leq c'$ and $d' \leq d$. By anti-reduction, it suffices to show

$$\begin{aligned} & (\text{let } x = \langle d'_i \leftarrow d'_i \rangle \text{raise } \varepsilon (\langle c'_i \prec c'_i \rangle V^l) \text{ in } \langle \sigma_2 \prec \sigma_1 \rangle E^l [x], \\ & \text{let } x = \langle d_i \leftarrow d_i \rangle \text{raise } \varepsilon (\langle c_i \prec c_i \rangle V^r) \text{ in } \langle \sigma'_2 \prec \sigma'_1 \rangle E^r [x]) \\ & \in (\blacktriangleright \mathcal{E}_k^{\sim} \llbracket \sigma'_2 \rrbracket)_k (\mathcal{V}^{\sim} \llbracket c \rrbracket), \end{aligned}$$

By congruence for Let, it suffices to show (1)

$$\begin{aligned}
& \langle d_i^l \Leftarrow d_i^r \rangle \text{raise } \varepsilon(\langle c_i^r \Leftarrow c_i^l \rangle V^l), \\
& \langle d_i^l \Leftarrow d_i^{rr} \rangle \text{raise } \varepsilon(\langle c_i^{rr} \Leftarrow c_i^l \rangle V^r) \\
& \in (\blacktriangleright \mathcal{E}_j^\sim \llbracket \sigma_2^\prime \rrbracket_k (\mathcal{V}^\sim \llbracket c \rrbracket)),
\end{aligned}$$

and (2) for $(V_1, V_2) \in \blacktriangleright (\mathcal{V}^\sim \llbracket d_i \rrbracket)_k$ we have

$$\begin{aligned}
& \langle \sigma_2 \Leftarrow \sigma_1 \rangle E^l[V_1], \\
& \langle \sigma_2^\prime \Leftarrow \sigma_1^\prime \rangle E^r[V_2] \\
& \in (\blacktriangleright \mathcal{E}_j^\sim \llbracket \sigma_2^\prime \rrbracket_k (\mathcal{V}^\sim \llbracket c \rrbracket)),
\end{aligned}$$

To show (1), first note that by the induction hypothesis for value types,

$$(\text{raise } \varepsilon(\langle c_i^r \Leftarrow c_i^l \rangle V^l), \text{raise } \varepsilon(\langle c_i^{rr} \Leftarrow c_i^l \rangle V^r)) \in \mathcal{E}_k^\sim \llbracket \sigma_2^\prime \rrbracket \mathcal{V}^\sim \llbracket c_i^\prime \rrbracket,$$

and by the induction hypothesis for value types again, (1) follows. To show (2), note that $E^l[x^l]$ and $E^r[x^r]$ are related by assumption on E^l and E^r . So we may apply the Löb induction hypothesis to reach the desired conclusion.

- (4) We again use Löb induction and monadic bind. In the related raises case of the bind lemma, we let $\varepsilon @ c_i \rightsquigarrow d_i \in \sigma_2$ be an effect caught by $\langle \sigma_2^\prime \Leftarrow \sigma_1^\prime \rangle \bullet$. We let $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^\sim \llbracket c_i^\prime \rrbracket)_k$, and let $(E^l, E^r) \in (\blacktriangleright \mathcal{K}^\sim \llbracket d_i^\prime \rrbracket)_k (\mathcal{E}^\sim \llbracket \sigma_1^\prime \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket)$.

We need to show

$$\begin{aligned}
& \langle \sigma_2 \Leftarrow \sigma_1 \rangle E^l[\text{raise } \varepsilon(V^l)], \langle \sigma_2^\prime \Leftarrow \sigma_1^\prime \rangle E^r[\text{raise } \varepsilon(V^r)] \\
& \in \mathcal{E}_k^\sim \llbracket \sigma_2^\prime \rrbracket \mathcal{V}^\sim \llbracket c \rrbracket.
\end{aligned}$$

If $\varepsilon \notin \sigma_1$, then both sides step to \mathcal{U} . Since \mathcal{U} is related to itself by ErrBot (Lemma D.47), we are finished by anti-reduction.

Otherwise, the proof proceeds analogously to that of the previous case, with upcasts and downcasts interchanged.

□

LEMMA D.62 (EFFECT CASTS COMMUTE WITH PURE FUNCTION VALUES). *Let E be an evaluation context such that (1) for all $\sigma, \Sigma \mid \Gamma \mid \bullet : (\sigma!A) \vdash_\sigma E : B$, and such that (2) $E \# \varepsilon$ for all $\varepsilon \in \Sigma$. Furthermore, suppose that (3) for all values V , there exists a value V' such that $E[V] \mapsto^* V'$.*

Let $\Sigma \mid \Gamma^\square \vDash_{\sigma_2} M \equiv N : A$.

Then $\Sigma \mid \Gamma^\square \vDash_{\sigma_1} E[\langle \sigma_1 \Leftarrow \sigma_2 \rangle M] \equiv \langle \sigma_1 \Leftarrow \sigma_2 \rangle E[N] : B$, and likewise for upcasts.

PROOF. We show the statement for downcasts only; the proof for upcasts is similar. Additionally, we show only one of the directions of the equivalence; the other is symmetric.

We need to show

$$(E[\langle \sigma_1 \Leftarrow \sigma_2 \rangle M], \langle \sigma_1 \Leftarrow \sigma_2 \rangle E[N]) \in \mathcal{E}_j^\sim \llbracket \sigma_1 \rrbracket \mathcal{V}^\sim \llbracket B \rrbracket.$$

We apply monadic bind (Lemma D.18) with $E_1 = E[\langle \sigma_1 \Leftarrow \sigma_2 \rangle \bullet]$ and $E_2 = \langle \sigma_1 \Leftarrow \sigma_2 \rangle E$. By assumption on M and N , will suffice to consider the following cases.

- Let $k \leq j$ and let $(V_1, V_2) \in \mathcal{V}_k^\sim \llbracket A \rrbracket$. We need to show

$$(E[\langle \sigma_1 \leftarrow \sigma_2 \rangle V_1], \langle \sigma_1 \leftarrow \sigma_2 \rangle E[V_2]) \in \mathcal{E}_k^\sim \llbracket \sigma_1 \rrbracket \mathcal{V}^\sim \llbracket B \rrbracket.$$

By the operational semantics, we have $E[\langle \sigma_1 \leftarrow \sigma_2 \rangle V_1] \mapsto^1 E[V_1]$.

By anti-reduction, it suffices to show

$$(E[V_1], \langle \sigma_1 \leftarrow \sigma_2 \rangle E[V_2]) \in \mathcal{E}_k^\sim \llbracket \sigma_1 \rrbracket \mathcal{V}^\sim \llbracket B \rrbracket.$$

Furthermore, there exist i_1 and i_2 and values V'_1 and V'_2 such that $E[V_1] \mapsto^{i_1} V'_1$ and $E[V_2] \mapsto^{i_2} V'_2$.

We also have $\langle \sigma_1 \leftarrow \sigma_2 \rangle V'_2 \mapsto^1 V'_2$.

Putting the above facts together, by anti-reduction, it suffices to show

$$(V'_1, V'_2) \in \mathcal{E}_k^\sim \llbracket \sigma_1 \rrbracket \mathcal{V}^\sim \llbracket B \rrbracket.$$

But by forward reduction, it suffices to show that $(E[V_1], E[V_2]) \in \mathcal{E}_k^\sim \llbracket \sigma_1 \rrbracket \mathcal{V}^\sim \llbracket B \rrbracket$.

For this, it suffices (by the congruence lemmas) that V_1 and V_2 are related, which is true by assumption.

- Let $k \leq j$ and let $\epsilon @ c^r \rightsquigarrow d^r \in \sigma_2$ be an effect caught by $\langle \sigma_1 \leftarrow \sigma_2 \rangle$. Let $V^l, V^r, E^l \# \epsilon, E^r \# \epsilon$ be as in the statement of Lemma D.18. We need to show

$$(E[\langle \sigma_1 \leftarrow \sigma_2 \rangle E^l[\text{raise } \epsilon(V^l)]], \langle \sigma_1 \leftarrow \sigma_2 \rangle E[E^r[\text{raise } \epsilon(V^r)]]) \in \mathcal{E}_k^\sim \llbracket \sigma_1 \rrbracket \mathcal{V}^\sim \llbracket B \rrbracket.$$

If $\epsilon \notin \sigma_1$, then, by the operational semantics, both terms will step to \mathcal{U} . By anti-reduction, it suffices to show that $(\mathcal{U}, \mathcal{U}) \in \mathcal{E}_k^\sim \llbracket \sigma_1 \rrbracket \mathcal{V}^\sim \llbracket B \rrbracket$. This follows by ErrBot (Lemma D.47).

Now suppose $\epsilon @ c^l \rightsquigarrow d^l \in \sigma_1$. According to the operational semantics, we have

$$E[\langle \sigma_1 \leftarrow \sigma_2 \rangle E^l[\text{raise } \epsilon @ c^r \rightsquigarrow d^r(V^l)]] \mapsto^1 E[E^l[\langle d^r \rightsquigarrow d^l \rangle \text{raise } \epsilon @ c^l \rightsquigarrow d^l(\langle c^l \leftarrow c^r \rangle V^l)]],$$

and

$$\langle \sigma_1 \leftarrow \sigma_2 \rangle E[E^r[\text{raise } \epsilon @ c^r \rightsquigarrow d^r(V^r)]] \mapsto^1 E[E^r[\langle d^r \rightsquigarrow d^l \rangle \text{raise } \epsilon @ c^l \rightsquigarrow d^l(\langle c^l \leftarrow c^r \rangle V^r)]].$$

Thus, by anti-reduction, it suffices to show

$$\begin{aligned} & (E[E^l[\langle d^r \rightsquigarrow d^l \rangle \text{raise } \epsilon @ c^l \rightsquigarrow d^l(\langle c^l \leftarrow c^r \rangle V^l)]], \\ & E[E^r[\langle d^r \rightsquigarrow d^l \rangle \text{raise } \epsilon @ c^l \rightsquigarrow d^l(\langle c^l \leftarrow c^r \rangle V^r)]]) \\ & \in \mathcal{E}_k^\sim \llbracket \sigma_1 \rrbracket \mathcal{V}^\sim \llbracket B \rrbracket. \end{aligned}$$

Let V^l be the value to which $\langle c^l \leftarrow c^r \rangle V^l$ steps, and similarly let V^r be the value to which $\langle c^l \leftarrow c^r \rangle V^r$ steps. By anti-reduction, it suffices to show

$$\begin{aligned} & (E[E^l[\langle d^r \rightsquigarrow d^l \rangle \text{raise } \epsilon @ c^l \rightsquigarrow d^l(V^l)]], \\ & E[E^r[\langle d^r \rightsquigarrow d^l \rangle \text{raise } \epsilon @ c^l \rightsquigarrow d^l(V^r)]]) \\ & \in \mathcal{E}_k^\sim \llbracket \sigma_1 \rrbracket \mathcal{V}^\sim \llbracket B \rrbracket. \end{aligned}$$

As neither term steps, it is sufficient to show that they are related in $\mathcal{R}_k^{\sim}[\![\sigma_1]\!] \mathcal{V}^{\sim}[\![B]\!]$. We assert the second disjunct in the definition of $\mathcal{R}^{\sim}[\![\cdot]\!]$, taking $E^l = E[E^l[\langle d^r \rightsquigarrow d^l \rangle \bullet]]$ and $E^r = E[E^r[\langle d^r \rightsquigarrow d^l \rangle \bullet]]$.

We first need to show that $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^{\sim}[\![c]\!])_k$. By forward reduction, it suffices to show that

$$(\langle c^l \leftarrow c^r \rangle V^l, \langle c^l \leftarrow c^r \rangle V^r) \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma_1]\!])_k (\mathcal{V}^{\sim}[\![c^r]\!]).$$

By monotonicity of casts (lemma D.65), it suffices to show $(V^l, V^r) \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma_1]\!])_k (\mathcal{V}^{\sim}[\![c^r]\!])$. This follows from our assumption about V^l and V^r .

We now need to show that

$$(x^l.E[E^l[\langle d^r \rightsquigarrow d^l \rangle x^l]], x^r.E[E^r[\langle d^r \rightsquigarrow d^l \rangle x^r]]) \in (\blacktriangleright \mathcal{K}^{\sim}[\![d]\!])_k (\mathcal{E}^{\sim}[\![\sigma_1]\!] \mathcal{V}^{\sim}[\![B]\!]).$$

To this end, let $k' \leq k$ and let $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^{\sim}[\![d^l]\!])_{k'}$. We need to show

$$(E[E^l[\langle d^r \rightsquigarrow d^l \rangle V_1]], E[E^r[\langle d^r \rightsquigarrow d^l \rangle V_2]]) \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma_1]\!])_{k'} (\mathcal{V}^{\sim}[\![B]\!]).$$

It will suffice by the soundness of the congruence rules to show that

$$(E^l[\langle d^r \rightsquigarrow d^l \rangle V_1], E^r[\langle d^r \rightsquigarrow d^l \rangle V_2]) \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma_1]\!])_{k'} (\mathcal{V}^{\sim}[\![B]\!]).$$

Let V'_1 and V'_2 be the values to which $\langle d^r \rightsquigarrow d^l \rangle V_1$ and $\langle d^r \rightsquigarrow d^l \rangle V_2$ step, respectively. By anti-reduction, it suffices to show

$$(E^l[V'_1], E^r[V'_2]) \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma_1]\!])_{k'} (\mathcal{V}^{\sim}[\![B]\!]).$$

By assumption on E^l and E^r , it suffices to show that $(V'_1, V'_2) \in (\blacktriangleright \mathcal{V}^{\sim}[\![d^r]\!])_{k'}$. By forward reduction, it suffices to show

$$(\langle d^r \rightsquigarrow d^l \rangle V_1, \langle d^r \rightsquigarrow d^l \rangle V_2) \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma_1]\!])_{k'} (\mathcal{V}^{\sim}[\![B]\!]).$$

By monotonicity of casts (lemma D.65), it suffices to show

$$(V_1, V_2) \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma_1]\!])_{k'} (\mathcal{V}^{\sim}[\![B]\!]).$$

This follows from our assumption on V_1 and V_2 . □

COROLLARY D.63 (COMMUTATIVITY OF CASTS). *Value casts commute with effect casts.*

PROOF. This follows from D.62, because $\langle B \rightsquigarrow A \rangle \bullet$ and $\langle A \leftarrow B \rangle \bullet$ satisfy the requirements in the lemma. □

LEMMA D.64 (FUNCTORIALITY OF CASTS). *Let M be a term such that $\Sigma \mid \Gamma \cdot \vdash_{\sigma} M : A$. Let $c : A \sqsubseteq B$ and $e : B \sqsubseteq C$. Let $d_{\sigma} : \sigma \sqsubseteq \sigma'$ and let $d'_{\sigma} : \sigma' \sqsubseteq \sigma''$*

Suppose $\Sigma \mid \Gamma \sqsubseteq \vDash_{\sigma} M \equiv N : A$. Then the following hold:

Identity properties: *Suppose $\Sigma \mid \Gamma \sqsubseteq \vDash_{\sigma} M \sqsubseteq \vDash N : A$. We have*

- (1) $\Sigma \mid \Gamma \vDash_{\sigma} \langle A \rightsquigarrow A \rangle M \equiv N : A$
- (2) $\Sigma \mid \Gamma \vDash_{\sigma} \langle A \leftarrow A \rangle M \equiv N : A$
- (3) $\Sigma \mid \Gamma \vDash_{\sigma} \langle \sigma \rightsquigarrow \sigma \rangle M \equiv N : A$
- (4) $\Sigma \mid \Gamma \vDash_{\sigma} \langle \sigma \leftarrow \sigma \rangle M \equiv N : A$

Composition properties: *Let $c : A \sqsubseteq B$ and $e : B \sqsubseteq C$. Let $d_{\sigma} : \sigma \sqsubseteq \sigma'$ and $d'_{\sigma} : \sigma' \sqsubseteq \sigma''$. Suppose $M \sqsubseteq \vDash N$. Then*

- (1) $\Sigma \mid \Gamma \vDash_{\sigma} \langle C \hookrightarrow A \rangle M \sqsubseteq \langle C \hookrightarrow B \rangle \langle B \hookrightarrow A \rangle N : C$
- (2) $\Sigma \mid \Gamma \vDash_{\sigma} \langle A \leftarrow C \rangle M \sqsubseteq \langle A \leftarrow B \rangle \langle B \leftarrow C \rangle N : A$
- (3) $\Sigma \mid \Gamma \vDash_{\sigma''} \langle \sigma'' \hookrightarrow \sigma \rangle M \sqsubseteq \langle \sigma'' \hookrightarrow \sigma' \rangle \langle \sigma' \hookrightarrow \sigma \rangle N : A$
- (4) $\Sigma \mid \Gamma \vDash_{\sigma} \langle \sigma \leftarrow \sigma'' \rangle M \sqsubseteq \langle \sigma \leftarrow \sigma' \rangle \langle \sigma' \leftarrow \sigma'' \rangle N : A$

PROOF. We prove more general, “pointwise” versions of the above statements. For instance, we show that if $(M, N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!]$, then $(\langle A \hookrightarrow A \rangle M, N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!]$.

Additionally, we only prove one direction of each of the equivalences (i.e., \sqsubseteq); the proof of the other direction is symmetric.

The statements are proven simultaneously by induction on A and σ .

- Identity properties:

- (1) We need to show $(\langle A \hookrightarrow A \rangle M, N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!]$. By monadic bind (Lemma D.18), with $E_1 = \langle A \hookrightarrow A \rangle \bullet$ and $E_2 = \bullet$, it will suffice to show the following: Let $k \leq j$ and $(V_1, V_2) \in \mathcal{V}_k^{\sim}[\![A]\!]$. We will show

$$(\langle A \hookrightarrow A \rangle V_1, V_2) \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!]$$

We continue by induction on A . If $A = \text{bool}$, then we need to show

$$(\langle \text{bool} \hookrightarrow \text{bool} \rangle V_1, V_2) \in \mathcal{E}_k^{\sim}[\![d_{\sigma}]\!] \mathcal{V}^{\sim}[\![\text{bool}]\!]$$

By anti-reduction, it suffices to show $(V_1, V_2) \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![\text{bool}]\!]$, which follows from our assumption on (V_1, V_2) .

If $A = A_i \rightarrow_{\sigma_A} A_o$, we need to show

$$(\langle (A_i \rightarrow_{\sigma_A} A_o) \hookrightarrow (A_i \rightarrow_{\sigma_A} A_o) \rangle V_1, V_2) \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A_i \rightarrow_{\sigma_A} A_o]\!]$$

As both terms are values, it suffices to show they are related in $\mathcal{V}_k^{\sim}[\![A_i \rightarrow_{\sigma_A} A_o]\!]$. So, let $k' \leq k$ and let $(V^l, V^r) \in \mathcal{V}_{k'}^{\sim}[\![A_i]\!]$. We need to show

$$(\langle (A_i \rightarrow_{\sigma_A} A_o) \hookrightarrow (A_i \rightarrow_{\sigma_A} A_o) \rangle V_1, V_2, V^l, V^r) \in \mathcal{E}_{k'}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_o]\!]$$

By anti-reduction, it suffices to show

$$(\langle A_o \hookrightarrow A_o \rangle \langle \sigma_A \hookrightarrow \sigma_A \rangle (V_1 \langle A_i \leftarrow A_i \rangle V^l, V_2, V^r)) \in \mathcal{E}_{k'}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_o]\!]$$

By the induction hypothesis (applied twice), it suffices to show

$$((V_1 \langle A_i \leftarrow A_i \rangle V^l, V_2, V^r) \in \mathcal{E}_{k'}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_o]\!]$$

By the soundness of function application, it suffices to show that $(V_1, V_2) \in \mathcal{V}_{k'}^{\sim}[\![A_i \rightarrow_{\sigma_A} A_o]\!]$ and $(\langle A_i \leftarrow A_i \rangle V^l, V^r) \in \mathcal{E}_{k'}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_i]\!]$. The former is true by assumption and downward closure ($k' \leq k$). The latter is true by inductive hypothesis, since V^l and V^r are related.

- (2) This is dual to the above.

- (3) We prove this statement by Löb induction (Lemma D.16). That is, assume for all $(M', N') \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma]\!]_j) (\mathcal{V}^{\sim}[\![A]\!])$, we have $(\langle \sigma \hookrightarrow \sigma \rangle M', N') \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma]\!]_j) (\mathcal{V}^{\sim}[\![A]\!])$. Let $(M, N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!]$. We need to show $(\langle \sigma \hookrightarrow \sigma \rangle M, N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!]$. By monadic bind (Lemma D.18), with $E_1 = \langle \sigma \hookrightarrow \sigma \rangle \bullet$ and $E_2 = \bullet$, it will suffice to consider the following cases.

- Let $k \leq j$ and let $(V_1, V_2) \in \mathcal{V}_k^{\sim}[[A]]$. We need to show

$$(\langle \sigma \rightsquigarrow \sigma \rangle V_1, V_2) \in \mathcal{E}_k^{\sim}[[\sigma]]\mathcal{V}^{\sim}[[c]].$$

Per the operational semantics, we have $\langle \sigma \rightsquigarrow \sigma \rangle V_1 \mapsto^1 V_1$, so by anti-reduction it suffices to show $(V_1, V_2) \in \mathcal{E}_k^{\sim}[[\sigma]]\mathcal{V}^{\sim}[[A]]$, which follows by the assumption that $(V_1, V_2) \in \mathcal{V}_k^{\sim}[[A]]$.

- Let $k \leq j$ and let $\epsilon @ c_\epsilon \rightsquigarrow d_\epsilon$ be an effect caught by $\langle \sigma \rightsquigarrow \sigma \rangle \bullet$ – i.e., $\epsilon @ c_\epsilon \rightsquigarrow d_\epsilon \in \sigma$. Note that, as σ is a reflexivity derivation, c_ϵ and d_ϵ are also reflexivity derivations, i.e., $c_\epsilon^l = c_\epsilon^r$ and likewise for d_ϵ . For simplicity, let $C = c_\epsilon^l$ and $D = d_\epsilon^l$. Let $V^l, V^r, E^l \# \epsilon, E^r \# \epsilon$ be as in the statement of Lemma D.18. We need to show

$$\begin{aligned} & (\langle \sigma \rightsquigarrow \sigma \rangle E^l[\text{raise } \epsilon @ C \rightsquigarrow D(V^l)], E^r[\text{raise } \epsilon @ C \rightsquigarrow D(V^r)]) \\ & \in \mathcal{E}_k^{\sim}[[\sigma]]\mathcal{V}^{\sim}[[A]]. \end{aligned}$$

According to the operational semantics, we have

$$\begin{aligned} & \langle \sigma \rightsquigarrow \sigma \rangle E^l[\text{raise } \epsilon @ C \rightsquigarrow D(V^l)] \mapsto^1 \\ & \text{let } x = \langle D \leftarrow D \rangle \text{raise } (\epsilon @ C \rightsquigarrow D)(\langle C \rightsquigarrow C \rangle V^l) \text{ in } \langle \sigma \rightsquigarrow \sigma \rangle E^l[x] \end{aligned}$$

So, by anti-reduction it suffices to show that

$$\begin{aligned} & (\text{let } x = \langle D \leftarrow D \rangle \text{raise } (\epsilon @ C \rightsquigarrow D)(\langle C \rightsquigarrow C \rangle V^l) \text{ in } \langle \sigma \rightsquigarrow \sigma \rangle E^l[x], \\ & E^r[\text{raise } \epsilon @ C \rightsquigarrow D(V^r)]) \\ & \in \mathcal{E}_k^{\sim}[[\sigma]]\mathcal{V}^{\sim}[[A]]. \end{aligned}$$

Let V'^l be the term to which $\langle C \rightsquigarrow C \rangle V^l$ steps. By anti-reduction, it suffices to show that

$$\begin{aligned} & (\text{let } x = \langle D \leftarrow D \rangle \text{raise } (\epsilon @ C \rightsquigarrow D)(V'^l) \text{ in } \langle \sigma \rightsquigarrow \sigma \rangle E^l[x], \\ & E^r[\text{raise } \epsilon @ C \rightsquigarrow D(V^r)]) \\ & \in \mathcal{E}_k^{\sim}[[\sigma]]\mathcal{V}^{\sim}[[A]]. \end{aligned}$$

The above terms do not step, so it suffices to show that they are related in $\mathcal{R}_k^{\sim}[[\sigma]]\mathcal{V}^{\sim}[[A]]$. To this end, we will first show that $(V'^l, V^r) \in (\blacktriangleright \mathcal{V}^{\sim}[[C]])_k$. By forward reduction, it suffices to show that $(\langle C \rightsquigarrow C \rangle V^l, V^r) \in (\blacktriangleright \mathcal{E}^{\sim}[[\mathcal{V}^{\sim}[[C]]]])_k$. By the induction hypothesis, it suffices to show that $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^{\sim}[[C]])_{(k)}$.

Now we will show that

$$\begin{aligned} & (x^l.(\text{let } x = \langle D \leftarrow D \rangle x^l \text{ in } \langle \sigma \rightsquigarrow \sigma \rangle E^l[x]), x^r.E^r[x^r]) \\ & \in (\blacktriangleright \mathcal{K}^{\sim}[[D]])_k(\mathcal{E}^{\sim}[[\sigma]]\mathcal{V}^{\sim}[[A]]). \end{aligned}$$

Let $k' \leq k$ and let $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^{\sim}[[A]])_{k'}$. We need to show

$$\begin{aligned} & ((\text{let } x = \langle D \leftarrow D \rangle V_1 \text{ in } \langle \sigma \rightsquigarrow \sigma \rangle E^l[x]), E^r[V_2]) \\ & \in (\blacktriangleright \mathcal{E}^{\sim}[[\sigma]])_{k'}(\mathcal{V}^{\sim}[[A]]). \end{aligned}$$

Let V'_1 be the value to which $\langle D \leftarrow D \rangle V_1$ steps. By anti-reduction, it suffices to show

$$\begin{aligned} & ((\text{let } x = V_1' \text{ in } \langle \sigma \frown \sigma \rangle E^l[x], E^r[V_2]) \\ & \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_{k'}(\mathcal{V}^\sim[A]), \end{aligned}$$

and then since the let term steps, it suffices by anti-reduction again to show

$$(\langle \sigma \frown \sigma \rangle E^l[V_1'], E^r[V_2]) \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_{k'}(\mathcal{V}^\sim[A]),$$

By the Löb induction hypothesis, it suffices to show that

$$(E^l[V_1'], E^r[V_2]) \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_{k'}(\mathcal{V}^\sim[A])$$

By our assumption on E^l and E^r , it suffices to show

$$(V_1', V_2) \in (\blacktriangleright \mathcal{V}^\sim[A])_{k'}.$$

By forward reduction, it suffices to show

$$(\langle D \leftarrow D \rangle V_1, V_2) \in (\blacktriangleright \mathcal{V}^\sim[A])_{k'}.$$

By the induction hypothesis for value types, it suffices to show

$$(V_1, V_2) \in (\blacktriangleright \mathcal{E}^\sim[A])_{k'}.$$

This follows by assumption.

- (4) We again use Löb induction and monadic bind.

That is, assume for all $(M', N') \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_j(\mathcal{V}^\sim[A])$, we have $(\langle \sigma \leftarrow \sigma \rangle M', N') \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_j(\mathcal{V}^\sim[A])$. We need to show

$$(\langle \sigma \leftarrow \sigma \rangle M, N) \in \mathcal{E}_j^\sim[\sigma] \mathcal{V}^\sim[A]$$

where $(M, N) \in \mathcal{E}_j^\sim[\sigma] \mathcal{V}^\sim[A]$. We again use monadic bind, and as in the previous proof, the case of related values follows trivially since effect casts are the identity on values. Thus, it will suffice to show the related raises case. That is, let $k \leq j$ and let $\varepsilon @ c_\varepsilon \rightsquigarrow d_\varepsilon$ be an effect caught by $\langle \sigma \leftarrow \sigma \rangle \bullet$ – i.e., $\varepsilon @ c_\varepsilon \rightsquigarrow d_\varepsilon \in \sigma$. As in the previous proof, since σ is a reflexivity derivation, c_ε and d_ε are also reflexivity derivations, so for simplicity, let $C = c_\varepsilon^l = c_\varepsilon^r$ and $D = d_\varepsilon^l = d_\varepsilon^r$.

Let $V^l, V^r, E^l \# \varepsilon, E^r \# \varepsilon$ be as in the statement of the monadic bind lemma. We need to show

$$\begin{aligned} & (\langle \sigma \leftarrow \sigma \rangle E^l[\text{raise } \varepsilon @ C \rightsquigarrow D(V^l)], E^r[\text{raise } \varepsilon @ C \rightsquigarrow D(V^r)]) \\ & \in \mathcal{E}_k^\sim[\sigma] \mathcal{V}^\sim[A]. \end{aligned}$$

Note that, since $\varepsilon \in \sigma$, the downcast cannot fail.

The remainder of the proof proceeds exactly like the previous proof, with upcasts and downcasts interchanged.

- Composition properties:

- (1) We need to show $(\langle C \frown A \rangle M, \langle C \frown B \rangle \langle B \frown A \rangle N) \in \mathcal{E}_j^\sim[\sigma] \mathcal{V}^\sim[C]$.

By monadic bind (Lemma D.18) with $E_1 = \langle C \frown A \rangle \bullet$ and $E_2 = \langle C \frown B \rangle \langle B \frown A \rangle \bullet$, it will suffice to show the following: Let $k \leq j$ and let $(V_1, V_2) \in \mathcal{V}_k^\sim[A]$. We will show

$$(\langle C \frown A \rangle V_1, \langle C \frown B \rangle \langle B \frown A \rangle V_2) \in \mathcal{E}_k^\sim[\sigma] \mathcal{V}^\sim[C].$$

If $c \circ e = \text{bool}$, then $c = e = \text{bool}$, and we need to show

$$\langle \langle \text{bool} \rightsquigarrow \text{bool} \rangle V_1, \langle \text{bool} \rightsquigarrow \text{bool} \rangle \langle \text{bool} \rightsquigarrow \text{bool} \rangle V_2 \rangle \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![\text{bool}]\!].$$

By anti-reduction, it suffices to show $(V_1, V_2) \in \mathcal{V}_j^{\sim}[\![\text{bool}]\!]$, which follows from our assumption.

Now suppose $c \circ e = (c_i \circ e_i) \rightarrow_{(c_o \circ e_o)} (c_o \circ e_o)$. We need to show

$$\begin{aligned} & \langle \langle (C_i \rightarrow_{\sigma_C} C_o) \rightsquigarrow (A_i \rightarrow_{\sigma_A} A_o) \rangle V_1, \\ & \langle \langle (C_i \rightarrow_{\sigma_C} C_o) \rightsquigarrow (B_i \rightarrow_{\sigma_B} B_o) \rangle \langle (B_i \rightarrow_{\sigma_B} B_o) \rightsquigarrow (A_i \rightarrow_{\sigma_A} A_o) \rangle V_2 \rangle \\ & \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![C_i \rightarrow_{\sigma_C} C_o]\!]. \end{aligned}$$

Both terms are values, so it suffices to show that they are related in $\mathcal{V}_k^{\sim}[\![C_i \rightarrow_{\sigma_C} C_o]\!]$. Let $k' \leq k$ and let $(V^l, V^r) \in \mathcal{V}_{k'}^{\sim}[\![C_i]\!]$. We need to show that

$$\begin{aligned} & \langle \langle \langle (C_i \rightarrow_{\sigma_C} C_o) \rightsquigarrow (A_i \rightarrow_{\sigma_A} A_o) \rangle V_1 \rangle V^l, \\ & \langle \langle (C_i \rightarrow_{\sigma_C} C_o) \rightsquigarrow (B_i \rightarrow_{\sigma_B} B_o) \rangle \langle (B_i \rightarrow_{\sigma_B} B_o) \rightsquigarrow (A_i \rightarrow_{\sigma_A} A_o) \rangle V_2 \rangle V^r \rangle \\ & \in \mathcal{E}_{k'}^{\sim}[\![\sigma_C]\!] \mathcal{V}^{\sim}[\![C_o]\!]. \end{aligned}$$

By anti-reduction, it suffices to show

$$\begin{aligned} & \langle \langle C_o \rightsquigarrow A_o \rangle \langle \sigma_C \rightsquigarrow \sigma_A \rangle (V_1 \langle A_i \rightsquigarrow C_i \rangle V^l), \\ & \langle C_o \rightsquigarrow B_o \rangle \langle \sigma_C \rightsquigarrow \sigma_B \rangle \\ & \langle \langle \langle (B_i \rightarrow_{\sigma_B} B_o) \rightsquigarrow (A_i \rightarrow_{\sigma_A} A_o) \rangle V_2 \rangle \langle B_i \rightsquigarrow C_i \rangle V^r \rangle \\ & \in \mathcal{E}_{k'}^{\sim}[\![\sigma_C]\!] \mathcal{V}^{\sim}[\![C_o]\!]. \end{aligned}$$

Let V'' be the value to which $\langle B_i \rightsquigarrow C_i \rangle V^r$ steps. By anti-reduction, it suffices to show

$$\begin{aligned} & \langle \langle C_o \rightsquigarrow A_o \rangle \langle \sigma_C \rightsquigarrow \sigma_A \rangle (V_1 \langle A_i \rightsquigarrow C_i \rangle V^l), \\ & \langle C_o \rightsquigarrow B_o \rangle \langle \sigma_C \rightsquigarrow \sigma_B \rangle \\ & \langle \langle \langle (B_i \rightarrow_{\sigma_B} B_o) \rightsquigarrow (A_i \rightarrow_{\sigma_A} A_o) \rangle V_2 \rangle V'' \rangle \\ & \in \mathcal{E}_{k'}^{\sim}[\![\sigma_C]\!] \mathcal{V}^{\sim}[\![C_o]\!]. \end{aligned}$$

By anti-reduction again, it suffices to show

$$\begin{aligned} & \langle \langle C_o \rightsquigarrow A_o \rangle \langle \sigma_C \rightsquigarrow \sigma_A \rangle (V_1 \langle A_i \rightsquigarrow C_i \rangle V^l), \\ & \langle C_o \rightsquigarrow B_o \rangle \langle \sigma_C \rightsquigarrow \sigma_B \rangle \\ & \langle \langle \langle B_o \rightsquigarrow A_o \rangle \langle \sigma_B \rightsquigarrow \sigma_A \rangle (V_2 \langle A_i \rightsquigarrow B_i \rangle V'') \rangle \rangle \\ & \in \mathcal{E}_{k'}^{\sim}[\![\sigma_C]\!] \mathcal{V}^{\sim}[\![C_o]\!]. \end{aligned}$$

We will appeal to transitivity (Lemma D.66). We continue by cases on \sim .

– First suppose $\sim = <$. We first claim that

$$\begin{aligned}
& (\langle C_o \frown A_o \rangle \langle \sigma_C \frown \sigma_A \rangle (V_1 \langle A_i \leftarrow C_i \rangle V^l), \\
& \langle C_o \frown B_o \rangle \langle B_o \frown A_o \rangle \\
& \quad (\langle \sigma_C \frown \sigma_B \rangle \langle \sigma_B \frown \sigma_A \rangle (V_2 \langle A_i \leftarrow B_i \rangle V^r))) \\
& \in \mathcal{E}_{k'}^{\sim} \llbracket \sigma_C \rrbracket \mathcal{V}^{\sim} \llbracket C_o \rrbracket.
\end{aligned}$$

By the induction hypothesis applied twice, it suffices to show

$$\begin{aligned}
& ((V_1 \langle A_i \leftarrow C_i \rangle V^l), (V_2 \langle A_i \leftarrow B_i \rangle V^r)) \\
& \in \mathcal{E}_{k'}^{\sim} \llbracket \sigma_A \rrbracket \mathcal{V}^{\sim} \llbracket A_o \rrbracket.
\end{aligned}$$

By soundness of function application, it suffices to show that $(V_1, V_2) \in \mathcal{V}_{k'}^{\sim} \llbracket A_i \rightarrow_{\sigma_A} A_o \rrbracket$ and that

$$(\langle A_i \leftarrow C_i \rangle V^l, \langle A_i \leftarrow B_i \rangle V^r) \in \mathcal{E}_{k'}^{\sim} \llbracket \sigma \rrbracket \mathcal{V}^{\sim} \llbracket A_i \rrbracket.$$

The former holds by assumption and downward closure. To show the latter, it suffices by forward reduction to show that

$$(\langle A_i \leftarrow C_i \rangle V^l, \langle A_i \leftarrow B_i \rangle \langle B_i \leftarrow C_i \rangle V^r) \in \mathcal{E}_{k'}^{\sim} \llbracket \sigma \rrbracket \mathcal{V}^{\sim} \llbracket A_i \rrbracket.$$

Now, by the induction hypothesis, it suffices to show that

$$(V^l, V^r) \in \mathcal{E}_{k'}^{\sim} \llbracket \sigma \rrbracket \mathcal{V}^{\sim} \llbracket C_i \rrbracket,$$

which follows from our assumption.

Now by transitivity, it will suffice to show

$$\begin{aligned}
& (\langle C_o \frown B_o \rangle \langle B_o \frown A_o \rangle \\
& \quad (\langle \sigma_C \frown \sigma_B \rangle \langle \sigma_B \frown \sigma_A \rangle (V_2 \langle A_i \leftarrow B_i \rangle V^r))), \\
& \langle C_o \frown B_o \rangle \langle \sigma_C \frown \sigma_B \rangle \\
& \quad (\langle B_o \frown A_o \rangle \langle \sigma_B \frown \sigma_A \rangle (V_2 \langle A_i \leftarrow B_i \rangle V^r))) \\
& \in \mathcal{E}_{\omega}^{\sim} \llbracket \sigma_C \rrbracket \mathcal{V}^{\sim} \llbracket C_o \rrbracket.
\end{aligned}$$

By reflexivity (Corollary D.30), we have that $\langle B_o \frown A_o \rangle \langle \sigma_C \frown \sigma_B \rangle \langle \sigma_B \frown \sigma_A \rangle (V_2 \langle A_i \leftarrow B_i \rangle V^r)$ is related to itself. Then by commutativity of casts (Corollary D.63), we can interchange the order of $\langle B_o \frown A_o \rangle$ and $\langle \sigma_C \frown \sigma_B \rangle$, and the resulting terms are related. Finally by monotonicity of casts (Lemma D.65), we can apply $\langle C_o \frown B_o \rangle$, and the resulting terms are still related. Moreover, all of these relations hold “at ω ”.

– Now suppose $\sim = >$. By similar reasoning as in the previous case, we have

$$\begin{aligned}
& (\langle C_o \frown A_o \rangle \langle \sigma_C \frown \sigma_A \rangle (V_1 \langle A_i \leftarrow C_i \rangle V^l), \\
& \langle C_o \frown B_o \rangle \langle B_o \frown A_o \rangle \\
& \quad (\langle \sigma_C \frown \sigma_B \rangle \langle \sigma_B \frown \sigma_A \rangle (V_2 \langle A_i \leftarrow B_i \rangle V^l))) \\
& \in \mathcal{E}_{\omega}^{\sim} \llbracket \sigma_C \rrbracket \mathcal{V}^{\sim} \llbracket C_o \rrbracket.
\end{aligned}$$

Thus, by transitivity it will suffice to show

$$\begin{aligned} & \langle \langle C_o \hookrightarrow B_o \rangle \langle B_o \hookrightarrow A_o \rangle \\ & \quad \langle \langle \sigma_C \hookrightarrow \sigma_B \rangle \langle \sigma_B \hookrightarrow \sigma_A \rangle (V_2 \langle A_i \leftarrow B_i \rangle V^l) \rangle, \\ & \langle C_o \hookrightarrow B_o \rangle \langle \sigma_C \hookrightarrow \sigma_B \rangle \\ & \quad \langle \langle B_o \hookrightarrow A_o \rangle \langle \sigma_B \hookrightarrow \sigma_A \rangle (V_2 \langle A_i \leftarrow B_i \rangle V^r) \rangle \rangle \\ & \in \mathcal{E}_{k'}^{\geq} \llbracket \sigma_C \rrbracket \mathcal{V}^{\sim} \llbracket C_o \rrbracket. \end{aligned}$$

The reasoning is analogous to that of the previous case.

(2) This is dual to the above.

(3) We prove this statement by Löb induction (Lemma D.16). That is, assume for all $(M', N') \in (\blacktriangleright \mathcal{E}^{\sim} \llbracket \sigma \rrbracket)_j (\mathcal{V}^{\sim} \llbracket A \rrbracket)$, we have

$$\langle \langle \sigma'' \hookrightarrow \sigma \rangle M, \langle \sigma'' \hookrightarrow \sigma' \rangle \langle \sigma' \hookrightarrow \sigma \rangle N \rangle \in (\blacktriangleright \mathcal{E}^{\sim} \llbracket \sigma'' \rrbracket)_j (\mathcal{V}^{\sim} \llbracket A \rrbracket).$$

Let $(M, N) \in \mathcal{E}_j^{\sim} \llbracket \sigma \rrbracket \mathcal{V}^{\sim} \llbracket A \rrbracket$. We need to show

$$\langle \langle \sigma'' \hookrightarrow \sigma \rangle M, \langle \sigma'' \hookrightarrow \sigma' \rangle \langle \sigma' \hookrightarrow \sigma \rangle N \rangle \in \mathcal{E}_j^{\sim} \llbracket \sigma'' \rrbracket \mathcal{V}^{\sim} \llbracket A \rrbracket.$$

By monadic bind (Lemma D.18), with $E_1 = \langle \sigma'' \hookrightarrow \sigma \rangle \bullet$ and $E_2 = \langle \sigma'' \hookrightarrow \sigma' \rangle \langle \sigma' \hookrightarrow \sigma \rangle \bullet$, it suffices to consider the following cases:

– Let $k \leq j$ and let $(V_1, V_2) \in \mathcal{V}_k^{\sim} \llbracket A \rrbracket$. We need to show that

$$\langle \langle \sigma'' \hookrightarrow \sigma \rangle V_1, \langle \sigma'' \hookrightarrow \sigma' \rangle \langle \sigma' \hookrightarrow \sigma \rangle V_2 \rangle \in \mathcal{E}_j^{\sim} \llbracket \sigma'' \rrbracket \mathcal{V}^{\sim} \llbracket A \rrbracket.$$

Since the effect cast is the identity on values, the above follows immediately by anti-reduction.

– Let $k \leq j$ and let $\varepsilon @ c_\varepsilon \rightsquigarrow d_\varepsilon \in \sigma$ be an effect caught by either E_1 or E_2 . Note that, as σ is a reflexivity derivation, c_ε and d_ε are also reflexivity derivations, i.e., $c_\varepsilon^l = c_\varepsilon^r$ and likewise for d_ε . For simplicity, let $C^L = c_\varepsilon^l$ and $D^L = d_\varepsilon^l$.

Let $V^l, V^r, E^l \# \varepsilon, E^r \# \varepsilon$ be as in the statement of the monadic bind lemma. We need to show

$$\begin{aligned} & \langle \langle \sigma'' \hookrightarrow \sigma \rangle E^l [\text{raise } (\varepsilon @ C^L \rightsquigarrow D^L) (V^l)], \\ & \langle \sigma'' \hookrightarrow \sigma' \rangle \langle \sigma' \hookrightarrow \sigma \rangle E^r [\text{raise } (\varepsilon @ C^L \rightsquigarrow D^L) (V^r)] \rangle \\ & \in \mathcal{E}_k^{\sim} \llbracket \sigma'' \rrbracket \mathcal{V}^{\sim} \llbracket A \rrbracket. \end{aligned}$$

Let C^M and D^M be the types such that $\varepsilon @ C^M \rightsquigarrow D^M \in \sigma'$. Let C^R and D^R be the types such that $\varepsilon @ C^R \rightsquigarrow D^R \in \sigma''$. By anti-reduction, it suffices to show

$$\begin{aligned} & (\text{let } x = \langle D^L \leftarrow D^R \rangle \text{raise } (\varepsilon @ C^R \rightsquigarrow D^R) (\langle C^R \hookrightarrow C^L \rangle V^l) \text{ in } \langle \sigma'' \hookrightarrow \sigma \rangle E^l [x], \\ & \langle \sigma'' \hookrightarrow \sigma' \rangle (\text{let } x = \langle D^L \leftarrow D^M \rangle \text{raise } (\varepsilon @ C^M \rightsquigarrow D^M) (\langle C^M \hookrightarrow C^L \rangle V^r) \text{ in } \langle \sigma' \hookrightarrow \sigma \rangle E^r [x]) \rangle \\ & \in \mathcal{E}_k^{\sim} \llbracket \sigma'' \rrbracket \mathcal{V}^{\sim} \llbracket A \rrbracket. \end{aligned}$$

Let V^l be the value to which $\langle C^R \hookrightarrow C^L \rangle V^l$ steps, say in i steps. Let V^r be the value to which $\langle C^M \hookrightarrow C^L \rangle V^r$ steps, say in j steps.

By anti-reduction, it suffices to show

$$\begin{aligned}
& (\text{let } x = \langle D^L \Leftarrow D^R \rangle \text{raise } (\epsilon @ C^R \rightsquigarrow D^R)(V^l) \text{ in } \langle \sigma'' \Leftarrow \sigma \rangle E^l[x], \\
& \langle \sigma'' \Leftarrow \sigma' \rangle (\text{let } x = \langle D^L \Leftarrow D^M \rangle \text{raise } (\epsilon @ C^M \rightsquigarrow D^M)(V^r) \text{ in } \langle \sigma' \Leftarrow \sigma \rangle E^r[x])) \\
& \in \mathcal{E}_k^{\sim}[\![\sigma'']\!] \mathcal{V}^{\sim}[\![A]\!].
\end{aligned}$$

Now (taking $E' = \text{let } x = \langle D^L \Leftarrow D^M \rangle \bullet \text{ in } \langle \sigma' \Leftarrow \sigma \rangle E^r[x]$ in the EFFUPCAST rule), it will suffice by anti-reduction to show

$$\begin{aligned}
& (\text{let } x = \langle D^L \Leftarrow D^R \rangle \text{raise } (\epsilon @ C^R \rightsquigarrow D^R)(V^l) \text{ in } \langle \sigma'' \Leftarrow \sigma \rangle E^l[x], \\
& \text{let } y = \langle D^M \Leftarrow D^R \rangle \text{raise } (\epsilon @ C^R \rightsquigarrow D^R)(\langle C^R \Leftarrow C^M \rangle V^r) \text{ in} \\
& \langle \sigma'' \Leftarrow \sigma' \rangle (\text{let } x = \langle D^L \Leftarrow D^M \rangle y \text{ in } \langle \sigma' \Leftarrow \sigma \rangle E^r[x])) \\
& \in \mathcal{E}_k^{\sim}[\![\sigma'']\!] \mathcal{V}^{\sim}[\![A]\!].
\end{aligned}$$

Let V''' be the value to which $\langle C^R \Leftarrow C^M \rangle V^r$ steps. By anti-reduction, it suffices to show

$$\begin{aligned}
& (\text{let } x = \langle D^L \Leftarrow D^R \rangle \text{raise } (\epsilon @ C^R \rightsquigarrow D^R)(V^l) \text{ in } \langle \sigma'' \Leftarrow \sigma \rangle E^l[x], \\
& \text{let } y = \langle D^M \Leftarrow D^R \rangle \text{raise } (\epsilon @ C^R \rightsquigarrow D^R)(V''') \text{ in} \\
& \langle \sigma'' \Leftarrow \sigma' \rangle (\text{let } x = \langle D^L \Leftarrow D^M \rangle y \text{ in } \langle \sigma' \Leftarrow \sigma \rangle E^r[x])) \\
& \in \mathcal{E}_k^{\sim}[\![\sigma'']\!] \mathcal{V}^{\sim}[\![A]\!].
\end{aligned}$$

As neither term steps, we will show that they belong to $\mathcal{R}_k^{\sim}[\![\sigma'']\!] \mathcal{V}^{\sim}[\![A]\!]$. We first need to show that

$$(V^l, V''') \in (\blacktriangleright \mathcal{V}^{\sim}[\![A]\!])_k.$$

By forward-reduction, it suffices to show that

$$(\langle C^R \Leftarrow C^L \rangle V^l, \langle C^R \Leftarrow C^M \rangle \langle C^M \Leftarrow C^L \rangle V^r) \in (\blacktriangleright \mathcal{V}^{\sim}[\![A]\!])_k.$$

By the induction hypothesis for value types, it suffices to show that $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^{\sim}[\![A]\!])_k$, which is true by assumption.

Now we need to show that, for all $k' \leq k$ and related values $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^{\sim}[\![A]\!])_{k'}$, we have

$$\begin{aligned}
& (\text{let } x = \langle D^L \Leftarrow D^R \rangle V_1 \text{ in } \langle \sigma'' \Leftarrow \sigma \rangle E^l[x], \\
& \text{let } y = \langle D^M \Leftarrow D^R \rangle V_2 \text{ in} \\
& \langle \sigma'' \Leftarrow \sigma' \rangle (\text{let } x = \langle D^L \Leftarrow D^M \rangle y \text{ in } \langle \sigma' \Leftarrow \sigma \rangle E^r[x])) \\
& \in (\blacktriangleright \mathcal{E}^{\sim}[\![\sigma'']\!])_{k'} (\mathcal{V}^{\sim}[\![A]\!]).
\end{aligned}$$

Let V'_1 and V'_2 be the values to which $\langle D^L \Leftarrow D^R \rangle V_1$ and $\langle D^M \Leftarrow D^R \rangle V_2$ step, respectively. By anti-reduction, it will suffice to show

$$\begin{aligned}
& \langle \sigma'' \prec_Y \sigma \rangle E^l[V'_1], \\
& \langle \sigma'' \prec_Y \sigma' \rangle (\text{let } x = \langle D^L \leftarrow D^M \rangle V'_2 \text{ in } \langle \sigma' \prec_Y \sigma \rangle E^r[x]) \\
& \in (\blacktriangleright \mathcal{E}^\sim[\sigma''])_{k'}(\mathcal{V}^\sim[A]).
\end{aligned}$$

Let V''_2 be the value to which $\langle D^L \leftarrow D^M \rangle V'_2$ steps. By anti-reduction, it will suffice to show

$$\begin{aligned}
& \langle \sigma'' \prec_Y \sigma \rangle E^l[V'_1], \\
& \langle \sigma'' \prec_Y \sigma' \rangle (\langle \sigma' \prec_Y \sigma \rangle E^r[V''_2]) \\
& \in (\blacktriangleright \mathcal{E}^\sim[\sigma''])_{k'}(\mathcal{V}^\sim[A]).
\end{aligned}$$

Now by the Löb induction hypothesis, it suffices to show

$$(E^l[V'_1], E^r[V''_2]) \in (\blacktriangleright \mathcal{E}^\sim[\sigma''])_{k'}(\mathcal{V}^\sim[A]).$$

By assumption on E^l and E^r , it suffices to show

$$(V'_1, V''_2) \in (\blacktriangleright \mathcal{V}^\sim[A])_{k'}.$$

Now by forward reduction it suffices to show

$$\langle D^L \leftarrow D^R \rangle V_1, \langle D^L \leftarrow D^M \rangle \langle D^M \leftarrow D^R \rangle V_2 \in (\blacktriangleright \mathcal{E}^\sim[\sigma''])_{k'}(\mathcal{V}^\sim[A]).$$

This follows by the inductive hypothesis for value types and our assumption on V_1 and V_2 .

- (4) This is dual to the above: we use Löb induction and monadic bind, and we reach a point where we need to show

$$\begin{aligned}
& \langle \sigma \leftarrow \sigma'' \rangle E^l[\text{raise } (\varepsilon @ C^R \rightsquigarrow D^R)(V^l)], \\
& \langle \sigma' \leftarrow \sigma'' \rangle \langle \sigma \leftarrow \sigma' \rangle E^r[\text{raise } (\varepsilon @ C^R \rightsquigarrow D^R)(V^r)] \\
& \in \mathcal{E}_k^\sim[\sigma](\mathcal{V}^\sim[A]).
\end{aligned}$$

where $\varepsilon @ C^R \rightsquigarrow D^R \in \sigma''$.

If $\varepsilon \notin \sigma$, then the left-hand side steps to \mathcal{U} , as does the right-hand side. By ErrBot (Lemma D.47), \mathcal{U} is related to itself, so by anti-reduction, we are finished. If $\varepsilon \notin \sigma'$, then in fact, $\varepsilon \notin \sigma$ (since $\sigma \sqsubseteq \sigma'$), and so again, both sides step to \mathcal{U} .

Otherwise, we proceed as in the proof of the previous case, with the upcasts and downcasts interchanged.

□

LEMMA D.65 (MONOTONICITY OF CASTS). *Let $c : A \sqsubseteq B$, and $d_\sigma : \sigma \sqsubseteq \sigma'$, and let M and N be terms such that $\Sigma \mid \Gamma^\sqsubseteq \vDash_\sigma M \sqsubseteq N : A$. The following hold:*

- (1) $\Sigma \mid \Gamma^\sqsubseteq \vDash_\sigma \langle B \prec_Y A \rangle M \sqsubseteq \langle B \prec_Y A \rangle N : B$
- (2) $\Sigma \mid \Gamma^\sqsubseteq \vDash_\sigma \langle A \leftarrow B \rangle M \sqsubseteq \langle A \leftarrow B \rangle N : A$
- (3) $\Sigma \mid \Gamma^\sqsubseteq \vDash_{\sigma'} \langle \sigma' \prec_Y \sigma \rangle M \sqsubseteq \langle \sigma' \prec_Y \sigma \rangle N : A$
- (4) $\Sigma \mid \Gamma^\sqsubseteq \vDash_\sigma \langle \sigma \leftarrow \sigma' \rangle M \sqsubseteq \langle \sigma \leftarrow \sigma' \rangle N : A$

PROOF. As in the proof of the functoriality properties of casts, we prove stronger, “pointwise” versions of the above statements, i.e., we assume $(M, N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A]\!]$, and show, for example, that $(\langle B \leftarrow A \rangle M, \langle B \leftarrow A \rangle N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![B]\!]$.

The proof is by induction on c and d_{σ} .

(1) We need to show

$$(\langle B \leftarrow A \rangle M, \langle B \leftarrow A \rangle N) \in \mathcal{E}_j^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![B]\!]$$

By monadic bind (Lemma D.18), with $E_1 = E_2 = \langle B \leftarrow A \rangle \bullet$, it will suffice to show that

$$(\langle B \leftarrow A \rangle V_1, \langle B \leftarrow A \rangle V_2) \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![B]\!]$$

where $k \leq j$ and let $(V_1, V_2) \in \mathcal{V}_k^{\sim}[\![A]\!]$.

If $c = \text{bool}$, then we need to show

$$(\langle \text{bool} \leftarrow \text{bool} \rangle V_1, \langle \text{bool} \leftarrow \text{bool} \rangle V_2) \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![\text{bool}]\!]$$

By anti-reduction, it suffices to show that $(V_1, V_2) \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![\text{bool}]\!]$, which follows from our assumption.

If $c = c_i \rightarrow_{c_{\sigma}} c_o$, then we need to show

$$\begin{aligned} & ((\langle B_i \rightarrow_{\sigma_B} B_o \rangle \leftarrow \langle A_i \rightarrow_{\sigma_A} A_o \rangle) V_1, (\langle B_i \rightarrow_{\sigma_B} B_o \rangle \leftarrow \langle A_i \rightarrow_{\sigma_A} A_o \rangle) V_2) \\ & \in \mathcal{E}_k^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![B_i \rightarrow_{\sigma_B} B_o]\!] \end{aligned}$$

As both terms are values, it suffices to show that they are related in $\mathcal{V}_k^{\sim}[\![B_i \rightarrow_{\sigma_B} B_o]\!]$. Let $k' \leq k$ and let $(V^l, V^r) \in \mathcal{K}_{k'}^{\sim}[\![B_i]\!]$. We need to show

$$\begin{aligned} & (((\langle B_i \rightarrow_{\sigma_B} B_o \rangle \leftarrow \langle A_i \rightarrow_{\sigma_A} A_o \rangle) V_1) V^l, \\ & ((\langle B_i \rightarrow_{\sigma_B} B_o \rangle \leftarrow \langle A_i \rightarrow_{\sigma_A} A_o \rangle) V_2) V^r) \\ & \in \mathcal{E}_{k'}^{\sim}[\![\sigma_B]\!] \mathcal{V}^{\sim}[\![B_o]\!] \end{aligned}$$

By anti-reduction, it suffices to show

$$\begin{aligned} & (\langle B_o \leftarrow A_o \rangle \langle \sigma_B \leftarrow \sigma_A \rangle (V_1 \langle A_i \leftarrow B_i \rangle V^l), \\ & \langle B_o \leftarrow A_o \rangle \langle \sigma_B \leftarrow \sigma_A \rangle (V_2 \langle A_i \leftarrow B_i \rangle V^r)) \\ & \in \mathcal{E}_{k'}^{\sim}[\![\sigma_B]\!] \mathcal{V}^{\sim}[\![B_o]\!] \end{aligned}$$

By the inductive hypothesis applied twice, it suffices to show

$$((V_1 \langle A_i \leftarrow B_i \rangle V^l), (V_2 \langle A_i \leftarrow B_i \rangle V^r)) \in \mathcal{E}_{k'}^{\sim}[\![\sigma_A]\!] \mathcal{V}^{\sim}[\![A_o]\!]$$

By soundness of function application, it suffices to show that $(V_1, V_2) \in \mathcal{V}_{k'}^{\sim}[\![A_i \rightarrow_{\sigma_A} A_o]\!]$ and that $(\langle A_i \leftarrow B_i \rangle V^l, \langle A_i \leftarrow B_i \rangle V^r) \in \mathcal{E}_{k'}^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![A_i]\!]$. The former is true by our assumption about V_1 and V_2 . To show the latter, it suffices by the inductive hypothesis to show that $(V^l, V^r) \in \mathcal{E}_{k'}^{\sim}[\![\sigma]\!] \mathcal{V}^{\sim}[\![B_i]\!]$, which follows by our assumption.

(2) This is dual to the above.

(3) This is dual to the below, and in fact easier since these are upcasts.

- (4) We prove this statement by Löb induction (Lemma D.16). That is, assume for all $(M', N') \in (\blacktriangleright \mathcal{E}^\sim[\sigma']_j(\mathcal{V}^\sim[A]))$, we have

$$(\langle \sigma \leftarrow \sigma' \rangle M, \langle \sigma \leftarrow \sigma' \rangle N) \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_j(\mathcal{V}^\sim[A]).$$

Let $(M, N) \in \mathcal{E}^\sim[\sigma']_j(\mathcal{V}^\sim[A])$. We need to show

$$(\langle \sigma \leftarrow \sigma' \rangle M, \langle \sigma \leftarrow \sigma' \rangle N) \in \mathcal{E}^\sim[\sigma]_j(\mathcal{V}^\sim[A]).$$

By monadic bind (Lemma D.18), it will suffice to consider the following two cases:

- Let $k \leq j$ and let $(V_1, V_2) \in \mathcal{V}^\sim_k[A]$. We need to show that

$$(\langle \sigma \leftarrow \sigma' \rangle V_1, \langle \sigma \leftarrow \sigma' \rangle V_2) \in \mathcal{E}^\sim[\sigma]_j(\mathcal{V}^\sim[A]).$$

Since the effect cast is the identity on values, the above follows immediately by anti-reduction.

- Let $k \leq j$ and let $\epsilon @ c_\epsilon \rightsquigarrow d_\epsilon \in \sigma'$ be an effect caught by $\langle \sigma \leftarrow \sigma' \rangle \bullet$. Recalling that σ' is shorthand for the reflexivity derivation $\sigma' \sqsubseteq \sigma'$, we have that c_ϵ and d_ϵ are themselves reflexivity (type precision) derivations; for brevity, we refer to the types as C and D . Let $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^\sim[C])_k$ and let $E^l \# \epsilon, E^r \# \epsilon$ be such that

$$(x^l.E^l[x^l], x^r.E^r[x^r]) \in (\blacktriangleright \mathcal{K}^\sim[D])_k(\mathcal{E}^\sim[\sigma]_j(\mathcal{V}^\sim[A])).$$

We need to show

$$\begin{aligned} & (\langle \sigma \leftarrow \sigma' \rangle E^l[\text{raise } \epsilon @ C \rightsquigarrow D(V^l)], \\ & \langle \sigma \leftarrow \sigma' \rangle E^r[\text{raise } \epsilon @ C \rightsquigarrow D(V^r)]) \\ & \in \mathcal{E}^\sim[\sigma]_k(\mathcal{V}^\sim[A]). \end{aligned}$$

First, if $\epsilon \notin \sigma$, then both sides step to \mathcal{U} , and we are finished by anti-reduction since \mathcal{U} is related to itself by ErrBot (Lemma D.47).

Otherwise, by anti-reduction, it suffices to show

$$\begin{aligned} & (\text{let } x = \langle D \rightsquigarrow C \rangle \text{raise } \epsilon @ C \rightsquigarrow D(\langle C \leftarrow C \rangle V^l) \text{ in } \langle \sigma \leftarrow \sigma' \rangle E^l[x], \\ & \text{let } x = \langle D \rightsquigarrow C \rangle \text{raise } \epsilon @ C \rightsquigarrow D(\langle C \leftarrow C \rangle V^r) \text{ in } \langle \sigma \leftarrow \sigma' \rangle E^r[x]) \\ & \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_k(\mathcal{V}^\sim[A]). \end{aligned}$$

By the soundness of the term precision congruence rule for let, it suffices to show that (1)

$$\begin{aligned} & (\langle D \rightsquigarrow C \rangle \text{raise } \epsilon @ C \rightsquigarrow D(\langle C \leftarrow C \rangle V^l), \\ & \langle D \rightsquigarrow C \rangle \text{raise } \epsilon @ C \rightsquigarrow D(\langle C \leftarrow C \rangle V^r)) \\ & \in (\blacktriangleright \mathcal{E}^\sim[\sigma])_k(\mathcal{V}^\sim[A]). \end{aligned}$$

and (2) for all related $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^\sim[A])$, we have

$$\begin{aligned} & (\langle \sigma \leftarrow \sigma' \rangle E^l[V_1], \langle \sigma \leftarrow \sigma' \rangle E^r[V_2]) \\ & \in \mathcal{E}^\sim[\sigma]_k(\mathcal{V}^\sim[A]). \end{aligned}$$

□

D.1.5 Transitivity. We introduce the following notation. We define $(M_1, M_2) \in R_\omega$ to mean that $(M_1, M_2) \in R_k$ for all natural numbers k .

We now state and prove a “mixed transitivity” lemma, in which we allow one of the two relations in the assumption to occur at a “proper” precision derivation, while the other is constrained to occur at a reflexivity derivation.

LEMMA D.66 (MIXED TRANSITIVITY, TERMS). *If (1) $(M_1, M_2) \in \mathcal{E}_\omega^\geq[\sigma]\mathcal{V}^\geq[A]$ and (2) $(M_2, M_3) \in \mathcal{E}_j^\geq[d_\sigma]\mathcal{V}^\geq[c]$, then $(M_1, M_3) \in \mathcal{E}_j^\geq[d_\sigma]\mathcal{V}^\geq[c]$.*

Similarly, if $(M_1, M_2) \in \mathcal{E}_j^\leq[d_\sigma]\mathcal{V}^\leq[c]$ and $(M_2, M_3) \in \mathcal{E}_\omega^\leq[\sigma]\mathcal{V}^\leq[A]$, then $(M_1, M_3) \in \mathcal{E}_j^\leq[d_\sigma]\mathcal{V}^\leq[c]$.

PROOF. This is proved simultaneously with the following two lemmas on transitivity for results and values. We prove the lemma for $\sim=>$; the other case is similar.

The proof is by Löb-induction (Lemma D.16). That is, assume that for all M'_1, M'_2 , and M'_3 , if $(M'_1, M'_2) \in (\blacktriangleright\mathcal{E}^\geq[\sigma])_\omega(\mathcal{V}^\geq[A])$ and $(M'_2, M'_3) \in (\blacktriangleright\mathcal{E}^\geq[d_\sigma])_j(\mathcal{V}^\geq[c])$, then $(M'_1, M'_3) \in (\blacktriangleright\mathcal{E}^\geq[d_\sigma])_j(\mathcal{V}^\geq[c])$.

We proceed by considering cases on the assumption that $(M_2, M_3) \in \mathcal{E}_j^\geq[d_\sigma]\mathcal{V}^\geq[c]$.

In the first case, $M_3 \mapsto^{j+1}$. Then we immediately have that $(M_1, M_3) \in \mathcal{E}_j^\geq[d_\sigma]\mathcal{V}^\geq[c]$, via the first disjunct.

In the second case, there is $k \leq j$ such that $M_3 \mapsto^{j-k} \mathcal{U}$ and $M_2 \mapsto^s \mathcal{U}$, for some number of steps s . By assumption (1), we have that $(M_1, M_2) \in \mathcal{E}_s^\geq[\sigma]\mathcal{V}^\geq[A]$. By inversion, we see that the second disjunct must have been true (with $k = 0$). This means in particular that $M_1 \mapsto^* \mathcal{U}$. Thus, we may conclude using the second disjunct that $(M_1, M_3) \in \mathcal{E}_j^\geq[d_\sigma]\mathcal{V}^\geq[c]$.

In the third case, there is $k \leq j$ and N_3 such that $M_3 \mapsto^{j-k} N_3$, and $M_2 \mapsto^s \mathcal{U}$, for some number of steps s . By similar reasoning to the previous case, we may conclude using the third disjunct that $(M_1, M_3) \in \mathcal{E}_j^\geq[d_\sigma]\mathcal{V}^\geq[c]$.

Finally, in the fourth case, there exist $k \leq j$ and $(N_2, N_3) \in \mathcal{R}_k^\geq[d_\sigma]\mathcal{V}^\geq[c]$ such that $M_2 \mapsto^s N_2$ for some s , and $M_3 \mapsto^{j-k} N_3$. By assumption (1), we have that $(M_1, M_2) \in \mathcal{E}_{s+i}^\geq[\sigma]\mathcal{V}^\geq[A]$ for all $i \in \mathbb{N}$. By inversion, we see that either the third or the fourth disjunct was true, with $k = i$ in both cases (notice that $(s+i) - i = s$, which is precisely the number of steps that M_2 takes to N_2).

In the former case, we have $M_1 \mapsto^* \mathcal{U}$ and we can then finish by asserting the third disjunct. In the latter case, there exists N_1 such that $M_1 \mapsto^* N_1$ and $(N_1, N_2) \in \mathcal{R}_i^\geq[\sigma]\mathcal{V}^\geq[A]$. Since i is arbitrary, this tells us that $(N_1, N_2) \in \mathcal{R}_\omega^\geq[\sigma]\mathcal{V}^\geq[A]$. To recap, we have $(N_1, N_2) \in \mathcal{R}_\omega^\geq[\sigma]\mathcal{V}^\geq[A]$, and $(N_2, N_3) \in \mathcal{R}_k^\geq[d_\sigma]\mathcal{V}^\geq[c]$, for some $k \leq j$. We want to show that $(N_1, N_3) \in \mathcal{R}_k^\geq[d_\sigma]\mathcal{V}^\geq[c]$.

This follows from Lemma D.68. □

LEMMA D.67 (MIXED TRANSITIVITY, VALUES). *If $(V_1, V_2) \in \mathcal{V}_\omega^\geq[A]$ and $(V_2, V_3) \in \mathcal{V}_j^\geq[c]$, then $(V_1, V_3) \in \mathcal{V}_j^\geq[c]$.*

Similarly, if $(V_1, V_2) \in \mathcal{V}_j^\leq[c]$ and $(V_2, V_3) \in \mathcal{V}_\omega^\leq[A]$, then $(V_1, V_3) \in \mathcal{V}_j^\leq[c]$.

PROOF. Proved simultaneously with the homogeneous transitivity for terms (Lemma D.66) and for results (Lemma D.68). The proof is by induction on the type precision derivation c . We prove the first statement only; the other is proved similarly.

- Case $c = \text{bool}$. Then we have $V_1 = V_2 = V_3$ and either all are true, or all are false. In either case, V_1 is related to V_3 .
- Case $c = c_i \rightarrow_{c_\sigma} c_o$. Then $A = A_i \rightarrow_{\sigma_A} A_o$ and $B = B_i \rightarrow_{\sigma_B} B_o$. We have $(V_1, V_2) \in \mathcal{V}_\omega^\sim[A_i \rightarrow_{\sigma_A} A_o]$ and $(V_2, V_3) \in \mathcal{V}_k^\sim[c_i \rightarrow_{c_\sigma} c_o]$.

We need to show

$$(V_1, V_3) \in \mathcal{V}_j^{\geq} \llbracket c_i \rightarrow_{c_\sigma} c_o \rrbracket.$$

Let $k \leq j$ and let $(V^l, V^r) \in \mathcal{V}_k^{\geq} \llbracket c_i \rrbracket$. We need to show that

$$(V_1 V^l, V_3 V^r) \in \mathcal{E}_k^{\geq} \llbracket c_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c_o \rrbracket.$$

By reflexivity (D.30), we know that $(V^l, V^l) \in \mathcal{V}_\omega^{\geq} \llbracket A_i \rrbracket$.

From our assumption about (V_1, V_2) , it follows that

$$(V_1 V^l, V_2 V^l) \in \mathcal{E}_\omega^{\geq} \llbracket \sigma_A \rrbracket \mathcal{V}^{\geq} \llbracket A_o \rrbracket.$$

From our assumption about (V_2, V_3) , we have

$$(V_2 V^l, V_3 V^r) \in \mathcal{E}_k^{\geq} \llbracket c_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c_o \rrbracket.$$

Now we apply the induction hypothesis (Lemma D.66) to conclude that

$$(V_1 V^l, V_3 V^r) \in \mathcal{E}_k^{\geq} \llbracket c_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c_o \rrbracket,$$

as needed. □

LEMMA D.68 (MIXED TRANSITIVITY, RESULTS). *If (1) $(N_1, N_2) \in \mathcal{R}_\omega^{\geq} \llbracket \sigma \rrbracket \mathcal{V}^{\geq} \llbracket A \rrbracket$ and (2) $(N_2, N_3) \in \mathcal{R}_j^{\geq} \llbracket d_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c \rrbracket$, then $(N_1, N_3) \in \mathcal{R}_j^{\geq} \llbracket d_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c \rrbracket$.*

Similarly, if $(N_1, N_2) \in \mathcal{R}_j^{\leq} \llbracket d_\sigma \rrbracket \mathcal{V}^{\leq} \llbracket c \rrbracket$ and $(N_2, N_3) \in \mathcal{R}_\omega^{\leq} \llbracket \sigma \rrbracket \mathcal{V}^{\leq} \llbracket A \rrbracket$, then $(N_1, N_3) \in \mathcal{R}_j^{\leq} \llbracket d_\sigma \rrbracket \mathcal{V}^{\leq} \llbracket c \rrbracket$.

PROOF. We prove only the first statement; the second is analogous.

Let j be fixed. We consider cases on assumption (1). There are two subcases to consider. First, N_1 and N_2 are values and $(N_1, N_2) \in \mathcal{V}_\omega^{\geq} \llbracket A \rrbracket$. Then N_3 is also a value, and $(N_2, N_3) \in \mathcal{V}_j^{\geq} \llbracket c \rrbracket$. By D.67, we have that $(N_1, N_3) \in \mathcal{V}_j^{\geq} \llbracket A \rrbracket$.

Otherwise, there exist $\epsilon @ C \rightsquigarrow D \in \sigma$, $E_1 \# \epsilon$ and $E_2 \# \epsilon$, and V_1 and V_2 such that $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^{\geq} \llbracket C \rrbracket)_\omega$, and $(x_1.E_1[x_1], x_2.E_2[x_2]) \in (\blacktriangleright \mathcal{K}^{\geq} \llbracket D \rrbracket)_\omega (\mathcal{E}^{\geq} \llbracket \sigma \rrbracket \mathcal{V}^{\geq} \llbracket A \rrbracket)$, and

$$N_1 = E_1[\text{raise } \epsilon(V_1)],$$

and

$$N_2 = E_2[\text{raise } \epsilon(V_2)].$$

Similarly, since N_2 and N_3 are related in $\mathcal{R}_j^{\geq} \llbracket d_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c \rrbracket$, it follows that $\epsilon @ c_\epsilon \rightsquigarrow d_\epsilon \in d_\sigma$, where $c_\epsilon : C \sqsubseteq C'$ and $d_\epsilon : D \sqsubseteq D'$. We also know that there exist $E_3 \# \epsilon$ and V_3 such that $(V_2, V_3) \in (\blacktriangleright \mathcal{V}^{\geq} \llbracket c_\epsilon \rrbracket)_j$, and $(x_2.E_2[x_2], x_3.E_3[x_3]) \in (\blacktriangleright \mathcal{K}^{\geq} \llbracket d_\epsilon \rrbracket)_j (\mathcal{E}^{\geq} \llbracket d_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c \rrbracket)$, and

$$N_3 = E_3[\text{raise } \epsilon(V_3)].$$

Recall that we need to show

$$(E_1[\text{raise } \epsilon(V_1)], E_3[\text{raise } \epsilon(V_3)]) \in \mathcal{R}_j^{\geq} \llbracket d_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c \rrbracket.$$

We assert the second disjunct in the definition of $\mathcal{R}^{\geq} \llbracket \cdot \rrbracket$.

We first claim that $(V_1, V_3) \in (\blacktriangleright \mathcal{V}^{\geq} \llbracket c_\epsilon \rrbracket)_j$. By transitivity for values (Lemma D.67), it suffices to show that $(V_1, V_2) \in (\blacktriangleright \mathcal{V}^{\geq} \llbracket c_\epsilon \rrbracket)_\omega$ and $(V_2, V_3) \in (\blacktriangleright \mathcal{V}^{\geq} \llbracket c_\epsilon \rrbracket)_j$. These follow by assumption.

Now we claim that

$$(x_1.E_1[x_1], x_3.E_3[x_3]) \in (\blacktriangleright \mathcal{K}^{\geq} \llbracket d_\epsilon \rrbracket)_j (\mathcal{E}^{\geq} \llbracket d_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c \rrbracket).$$

Let $k \leq j$ and let $(V^l, V^r) \in (\blacktriangleright \mathcal{V}^{\geq} \llbracket d_\epsilon \rrbracket)_k$. We need to show

$$(E_1[V^l], E_3[V^r]) \in (\blacktriangleright \mathcal{E}^{\geq} \llbracket d_\sigma \rrbracket)_k (\mathcal{V}^{\geq} \llbracket c \rrbracket).$$

By the induction hypothesis (recall we are proving this simultaneously with transitivity for terms, which is being proven by Löb induction), it suffices to find a term M such that $(E_1[V^l], M) \in (\blacktriangleright \mathcal{E}^{\geq} \llbracket \sigma \rrbracket)_\omega (\mathcal{V}^{\geq} \llbracket A \rrbracket)$, and $(M, E_3[V^r]) \in (\blacktriangleright \mathcal{E}^{\geq} \llbracket d_\sigma \rrbracket)_k (\mathcal{V}^{\geq} \llbracket c \rrbracket)$.

By reflexivity (Corollary D.30), we have $(V^l, V^l) \in (\blacktriangleright \mathcal{V}^{\sim} \llbracket \rrbracket)_\omega$.

Then by our assumption on (E_1, E_2) , we have

$$(E_1[V^l], E_2[V^l]) \in (\blacktriangleright \mathcal{E}^{\geq} \llbracket \sigma \rrbracket)_\omega (\mathcal{V}^{\geq} \llbracket A \rrbracket)$$

By our assumption on (E_2, E_3) we have

$$(E_2[V^l], E_3[V^r]) \in (\blacktriangleright \mathcal{E}^{\geq} \llbracket d_\sigma \rrbracket)_k (\mathcal{V}^{\geq} \llbracket c \rrbracket),$$

which finishes the proof. \square

LEMMA D.69 (HETEROGENEOUS TRANSITIVITY). *Let $c : A_1 \sqsubseteq A_2$ and $e : A_2 \sqsubseteq A_3$. Let $d_\sigma : \sigma \sqsubseteq \sigma'$ and let $d'_\sigma : \sigma' \sqsubseteq \sigma''$.*

If (1) $(M_1, M_2) \in \mathcal{E}_\omega^{\geq} \llbracket d_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c \rrbracket$ and (2) $(M_2, M_3) \in \mathcal{E}_j^{\geq} \llbracket d'_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket e \rrbracket$, then $(M_1, M_3) \in \mathcal{E}_j^{\geq} \llbracket d_\sigma \circ d'_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c \circ e \rrbracket$.

Similarly, if $(M_1, M_2) \in \mathcal{E}_j^{\leq} \llbracket d_\sigma \rrbracket \mathcal{V}^{\leq} \llbracket c \rrbracket$ and $(M_2, M_3) \in \mathcal{E}_\omega^{\leq} \llbracket d'_\sigma \rrbracket \mathcal{V}^{\leq} \llbracket e \rrbracket$, then $(M_1, M_3) \in \mathcal{E}_j^{\leq} \llbracket d_\sigma \circ d'_\sigma \rrbracket \mathcal{V}^{\leq} \llbracket c \circ e \rrbracket$.

PROOF. Follows from mixed transitivity (Lemma D.66) and the generalized cast lemmas (Lemmas D.50, D.51, D.52, D.53, D.54, D.55, D.56, and D.57).

For example, by EffDnR and ValDnR, we have

$$(M_1, \langle \sigma \leftarrow \sigma' \rangle \langle A_1 \leftarrow A_2 \rangle M_2) \in \mathcal{E}_\omega^{\geq} \llbracket \sigma \rrbracket \mathcal{V}^{\geq} \llbracket A_1 \rrbracket,$$

and by EffDnL and ValDnL, we have

$$(\langle \sigma \leftarrow \sigma' \rangle \langle A_1 \leftarrow A_2 \rangle M_2, M_3) \in \mathcal{E}_j^{\geq} \llbracket d_\sigma \circ d'_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c \circ e \rrbracket.$$

Then applying mixed transitivity, we have

$$(M_1, M_3) \in \mathcal{E}_j^{\geq} \llbracket d_\sigma \circ d'_\sigma \rrbracket \mathcal{V}^{\geq} \llbracket c \circ e \rrbracket,$$

as desired. \square