Fully Abstract Compilation via Universal Embedding

(Technical Appendix)

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Contents

1 Source Language λ^{S}

Types	σ	$::=$	$\alpha \mid 1 \mid \sigma_1 + \sigma_2 \mid \sigma_1 \times \sigma_2 \mid \sigma_1 \to \sigma_2 \mid \mu\alpha.\,\sigma$
Values	v	$::=$	$\mathsf{x} \mid \langle\rangle \mid \mathsf{inj}_1\,\mathsf{v} \mid \mathsf{inj}_2\,\mathsf{v} \mid \langle \mathsf{v}_1, \mathsf{v}_2\rangle \mid \lambda(\mathsf{x}:\sigma).\,\mathsf{e} \mid \mathsf{fold}_{\mu\alpha.\sigma}\,\mathsf{v}$
Expressions	e	$::=$	$\mathsf{v} \mid \mathsf{case}\,\mathsf{v}\,\mathsf{of}\,\mathsf{x}_1.\,\mathsf{e}_1 \mid \mathsf{x}_2.\,\mathsf{e}_2 \mid \pi_1\,\mathsf{v} \mid \pi_2\,\mathsf{v}\mathsf{v}_1\ \mathsf{v}_2 \mid \mathsf{unfold}\,\mathsf{v} \mid \mid \mathsf{let}\,\mathsf{x} = \mathsf{e}_1\,\mathsf{in}\,\mathsf{e}_2$
Eval. Contexts	K	$::=$	$[\cdot] \mid \mathsf{let}\,\mathsf{x} = \mathsf{K}\,\mathsf{in}\,\mathsf{e}_2$

Figure 1: Source Language (STLC): Syntax

Value Environment	Γ	$::=$	$\cdot \mid \Gamma, \mathsf{x} : \sigma$
Type Environment	Δ	$::=$	$\cdot \mid \Delta, \alpha$

$\boxed{\Delta \vdash \sigma}$

$$\frac{\alpha \in \Delta}{\Delta \vdash \alpha} \qquad \frac{}{\Delta \vdash 1} \qquad \frac{\Delta \vdash \sigma_1 \quad \Delta \vdash \sigma_2}{\Delta \vdash \sigma_1 + \sigma_2} \qquad \frac{\Delta \vdash \sigma_1 \quad \Delta \vdash \sigma_2}{\Delta \vdash \sigma_1 \times \sigma_2} \qquad \frac{\Delta \vdash \sigma_1 \quad \Delta \vdash \sigma_2}{\Delta \vdash \sigma_1 \to \sigma_2} \qquad \frac{\Delta, \alpha \vdash \sigma}{\Delta \vdash \mu\alpha.\,\sigma}$$

$\boxed{\Delta \vdash \Gamma}$

$$\frac{}{\Delta \vdash \cdot} \qquad \frac{\Delta \vdash \Gamma \quad \Delta \vdash \sigma}{\Delta \vdash \Gamma, \mathsf{x} : \sigma}$$

$\boxed{\Gamma \vdash \mathsf{e} : \sigma}$

$$\frac{\mathsf{x} : \sigma \in \Gamma \quad \cdot \vdash \Gamma}{\Gamma \vdash \mathsf{x} : \sigma} \qquad \frac{\cdot \vdash \Gamma}{\Gamma \vdash \langle\rangle : 1} \qquad \frac{\Gamma \vdash \mathsf{e} : \sigma_1 \quad \cdot \vdash \sigma_2}{\Gamma \vdash \mathsf{inj}_1\,\mathsf{e} : \sigma_1 + \sigma_2} \qquad \frac{\Gamma \vdash \mathsf{e} : \sigma_2 \quad \cdot \vdash \sigma_1}{\Gamma \vdash \mathsf{inj}_2\,\mathsf{e} : \sigma_1 + \sigma_2}$$

$$\frac{\Gamma \vdash \mathsf{v} : \sigma_1 + \sigma_2 \quad \Gamma, \mathsf{x}_1 : \sigma_1 \vdash \mathsf{e}_1 : \sigma \quad \Gamma, \mathsf{x}_2 : \sigma_2 \vdash \mathsf{e}_2 : \sigma}{\Gamma \vdash \mathsf{case}\,\mathsf{v}\,\mathsf{of}\,\mathsf{x}_1.\,\mathsf{e}_1 \mid \mathsf{x}_2.\,\mathsf{e}_2 : \sigma} \qquad \frac{\Gamma \vdash \mathsf{v}_1 : \sigma_1 \quad \Gamma \vdash \mathsf{v}_2 : \sigma_2}{\Gamma \vdash \langle \mathsf{v}_1, \mathsf{v}_2 \rangle : \sigma_1 \times \sigma_2} \qquad \frac{\Gamma \vdash \mathsf{v} : \sigma_1 \times \sigma_2}{\Gamma \vdash \pi_1\ : \sigma_1}$$

$$\frac{\Gamma \vdash \mathsf{v} : \sigma_1 \times \sigma_2}{\Gamma \vdash \pi_2\ : \sigma_2} \qquad \frac{\Gamma, \mathsf{x} : \sigma_1 \vdash \mathsf{e} : \sigma_2}{\Gamma \vdash \lambda(\mathsf{x}:\sigma_1).\,\mathsf{e} : \sigma_1 \to \sigma_2} \qquad \frac{\Gamma \vdash \mathsf{v}_1 : \sigma_2 \to \sigma \quad \Gamma \vdash \mathsf{v}_2 : \sigma_2}{\Gamma \vdash \mathsf{v}_1\ \mathsf{v}_2 : \sigma} \qquad \frac{\Gamma \vdash \mathsf{v} : \sigma[\mu\alpha.\,\sigma/\alpha]}{\Gamma \vdash \mathsf{fold}_{\mu\alpha.\sigma}\,\mathsf{v} : \mu\alpha.\,\sigma}$$

$$\frac{\Gamma \vdash \mathsf{v} : \mu\alpha.\,\sigma}{\Gamma \vdash \mathsf{unfold}\,\mathsf{v} : \sigma[\mu\alpha.\,\sigma/\alpha]} \qquad \frac{\Gamma \vdash \mathsf{e}_1 : \sigma_1 \quad \Gamma, \mathsf{x} : \sigma_1 \vdash \mathsf{e}_2 : \sigma_2}{\Gamma \vdash \mathsf{let}\,\mathsf{x} = \mathsf{e}_1\,\mathsf{in}\,\mathsf{e}_2 : \sigma_2}$$

Figure 2: Source Language (STLC): Static Semantics

$$
\begin{array}{rcl}
\mathsf{case}\,(\mathsf{inj}_1\,\mathsf{v})\,\mathsf{of}\,\mathsf{x}_1.\,\mathsf{e}_1 \mid \mathsf{x}_2.\,\mathsf{e}_2 & \overset{\mathsf{S}}{\hookrightarrow} & \mathsf{e}_1[\mathsf{v}/\mathsf{x}_1] \\
\mathsf{case}\,(\mathsf{inj}_2\,\mathsf{v})\,\mathsf{of}\,\mathsf{x}_1.\,\mathsf{e}_1 \mid \mathsf{x}_2.\,\mathsf{e}_2 & \overset{\mathsf{S}}{\hookrightarrow} & \mathsf{e}_2[\mathsf{v}/\mathsf{x}_2] \\
\pi_1\,\langle \mathsf{v}_1, \mathsf{v}_2 \rangle & \overset{\mathsf{S}}{\hookrightarrow} & \mathsf{v}_1 \\
\pi_2\,\langle \mathsf{v}_1, \mathsf{v}_2 \rangle & \overset{\mathsf{S}}{\hookrightarrow} & \mathsf{v}_2 \\
(\lambda(\mathsf{x} : \sigma).\,\mathsf{e})\ \mathsf{v} & \overset{\mathsf{S}}{\hookrightarrow} & \mathsf{e}[\mathsf{v}/\mathsf{x}] \\
\mathsf{unfold}\,(\mathsf{fold}_{\mu\alpha.\sigma}\,\mathsf{v}) & \overset{\mathsf{S}}{\hookrightarrow} & \mathsf{v} \\
\mathsf{let}\,\mathsf{x} = \mathsf{v}\,\mathsf{in}\,\mathsf{e} & \overset{\mathsf{S}}{\hookrightarrow} & \mathsf{e}[\mathsf{v}/\mathsf{x}]
\end{array}
$$

$$
\frac{\mathsf{e} \overset{\mathsf{S}}{\hookrightarrow} \mathsf{e}'}{\mathsf{K}[\mathsf{e}] \longmapsto \mathsf{K}[\mathsf{e}']}
$$

Figure 3: Source (λ^{S}): Operational Semantics

2 Target Language λ^{T}

Value Types	τ	$::=$	$\alpha \mid \tau_1 + \tau_2 \mid \langle\overline{\tau}\rangle \mid \forall[\alpha].\,\tau \to \theta \mid \mu\alpha.\,\tau \mid \exists\alpha.\,\tau \mid \mathbf{0}$
Computation Types	θ	$::=$	$\mathbf{E}\,\tau_1\,\tau_2$
Values	$\mathbf{v}$	$::=$	$\mathbf{x} \mid \mathbf{inj}_1\,\mathbf{v}_1 \mid \mathbf{inj}_2\,\mathbf{v}_2 \mid \langle\overline{\mathbf{v}}\rangle \mid \lambda[\alpha](\mathbf{x}:\tau).\,\mathbf{e} \mid \mathbf{fold}_{\mu\alpha.\tau}\,\mathbf{v} \mid \mathbf{pack}\,(\tau,\mathbf{v})\,\mathbf{as}\,\exists\alpha.\,\tau'$
Results	$\mathbf{r}$	$::=$	$\mathbf{return}\ \mathbf{v} \mid \mathbf{raise}\ \mathbf{v}$
Computations	$\mathbf{e}$	$::=$	$\mathbf{r} \mid \mathbf{v.i} \mid \mathbf{unfold}\,\mathbf{v} \mid \mathbf{handle}\,\mathbf{e}\,\mathbf{with}\,(\mathbf{x}.\,\mathbf{e}_1)\,(\mathbf{y}.\,\mathbf{e}_2) \mid \mathbf{v}_1\,[\tau]\,\mathbf{v}_2$ $\mid \mathbf{case}\,\mathbf{v}\,\mathbf{of}\,\mathbf{x}_1.\,\mathbf{e}_1 \mid \mathbf{x}_2.\,\mathbf{e}_2 \mid \mathbf{unpack}\,(\alpha,\mathbf{x}) = \mathbf{v}\,\mathbf{in}\,\mathbf{e}$
Evaluation Contexts	$\mathbf{K}$	$::=$	$[\cdot] \mid \mathbf{handle}\,\mathbf{K}\,\mathbf{with}\,(\mathbf{x}.\,\mathbf{e}_1)\,(\mathbf{y}.\,\mathbf{e}_2)$

$\boxed{\mathbf{e} \overset{\mathrm{T}}{\hookrightarrow} \mathbf{e}'}$

$$
\begin{array}{rcl}
\mathbf{case}\,(\mathbf{inj}_1\ \mathbf{v})\,\mathbf{of}\,\mathbf{x}_1.\,\mathbf{e}_1 \mid \mathbf{x}_2.\,\mathbf{e}_2 & \overset{\mathrm{T}}{\hookrightarrow} & \mathbf{e}_1[\mathbf{v}/\mathbf{x}_1] \\
\mathbf{case}\,(\mathbf{inj}_2\ \mathbf{v})\,\mathbf{of}\,\mathbf{x}_1.\,\mathbf{e}_1 \mid \mathbf{x}_2.\,\mathbf{e}_2 & \overset{\mathrm{T}}{\hookrightarrow} & \mathbf{e}_2[\mathbf{v}/\mathbf{x}_2] \\
\langle \mathbf{v}_1,\ldots,\mathbf{v}_n\rangle.\mathbf{i} & \overset{\mathrm{T}}{\hookrightarrow} & \mathbf{return}\ \mathbf{v}_i \\
(\lambda[\alpha](\mathbf{x}:\tau).\,\mathbf{e})\ [\tau']\ \mathbf{v} & \overset{\mathrm{T}}{\hookrightarrow} & \mathbf{e}[\tau'/\alpha][\mathbf{v}/\mathbf{x}] \\
\mathbf{unfold}\,(\mathbf{fold}_{\mu\alpha.\tau}\ \mathbf{v}) & \overset{\mathrm{T}}{\hookrightarrow} & \mathbf{return}\ \mathbf{v} \\
\mathbf{unpack}\,(\alpha,\mathbf{x}) = (\mathbf{pack}\,(\tau,\mathbf{v})\,\mathbf{as}\,\exists\alpha.\,\tau)\,\mathbf{in}\,\mathbf{e} & \overset{\mathrm{T}}{\hookrightarrow} & \mathbf{e}[\tau/\alpha][\mathbf{v}/\mathbf{x}] \\
\mathbf{handle}\,(\mathbf{return}\ \mathbf{v})\ \mathbf{with}\ (\mathbf{x}.\,\mathbf{e}_1)\ (\mathbf{y}.\,\mathbf{e}_2) & \overset{\mathrm{T}}{\hookrightarrow} & \mathbf{e}_1[\mathbf{v}/\mathbf{x}] \\
\mathbf{handle}\,(\mathbf{raise}\ \mathbf{v})\ \mathbf{with}\ (\mathbf{x}.\,\mathbf{e}_1)\ (\mathbf{y}.\,\mathbf{e}_2) & \overset{\mathrm{T}}{\hookrightarrow} & \mathbf{e}_2[\mathbf{v}/\mathbf{y}]
\end{array}
$$

$$
\frac{\mathbf{e} \overset{\mathrm{T}}{\hookrightarrow} \mathbf{e}'}{\mathbf{K}[\mathbf{e}] \longmapsto \mathbf{K}[\mathbf{e}']}
$$

Figure 4: Target Language (System F + exceptions): Syntax and Operational Semantics

$$
\begin{array}{llll}
\textit{Type Context} & \boldsymbol{\Delta} & ::= & \cdot \mid \boldsymbol{\Delta}, \boldsymbol{\alpha} \\
\textit{Value Context} & \boldsymbol{\Gamma} & ::= & \cdot \mid \boldsymbol{\Gamma}, \mathbf{x} : \boldsymbol{\tau}
\end{array}
$$

$\boxed{\Delta \vdash \tau}$

$$
\frac{\alpha \in \Delta}{\Delta \vdash \alpha} \qquad \frac{\Delta \vdash \tau_1 \qquad \Delta \vdash \tau_2}{\Delta \vdash \tau_1 + \tau_2} \qquad \frac{\Delta \vdash \tau_1 \cdots \Delta \vdash \tau_n}{\Delta \vdash \langle \tau_1, \ldots, \tau_n \rangle} \qquad \frac{\Delta, \alpha \vdash \tau_1 \qquad \Delta, \alpha \vdash \tau_2}{\Delta \vdash \forall[\alpha].\, \tau_1 \to \tau_2} \qquad \frac{\Delta, \alpha \vdash \tau}{\Delta \vdash \mu\alpha.\, \tau} \qquad \frac{}{\Delta \vdash \mathbf{0}}
$$

$$
\frac{\Delta, \alpha \vdash \tau}{\Delta \vdash \exists \alpha.\, \tau}
$$

$\boxed{\Delta \vdash \Gamma}$

$$
\frac{}{\Delta \vdash \cdot} \qquad \frac{\Delta \vdash \Gamma \qquad \Delta \vdash \tau}{\Delta \vdash \Gamma, \mathbf{x} : \tau}
$$

$\boxed{\Delta; \Gamma \vdash \mathbf{v} : \tau}$

$$
\frac{\Delta \vdash \Gamma \qquad \mathbf{x} : \tau \in \Gamma}{\Delta; \Gamma \vdash \mathbf{x} : \tau} \qquad \frac{\Delta; \Gamma \vdash \mathbf{v} : \tau_1 \qquad \Delta \vdash \tau_2}{\Delta; \Gamma \vdash \mathbf{inj_1}\, \mathbf{v} : \tau_1 + \tau_2} \qquad \frac{\Delta; \Gamma \vdash \mathbf{v} : \tau_2 \qquad \Delta \vdash \tau_1}{\Delta; \Gamma \vdash \mathbf{inj_2}\, \mathbf{v} : \tau_1 + \tau_2} \qquad \frac{\Delta; \Gamma \vdash \mathbf{v_i} : \tau_i}{\Delta; \Gamma \vdash \langle \overline{\mathbf{v}} \rangle : \langle \overline{\tau} \rangle}
$$

$$
\frac{\Delta \vdash \Gamma \qquad \alpha; \mathbf{x} : \tau \vdash \mathbf{e} : \theta}{\Delta; \Gamma \vdash \lambda[\alpha](\mathbf{x} : \tau).\, \mathbf{e} : \forall[\alpha].\, \tau \to \theta} \qquad \frac{\Delta; \Gamma \vdash \mathbf{v} : \tau[\mu\alpha.\, \tau/\alpha]}{\Delta; \Gamma \vdash \mathbf{fold}_{\mu\alpha.\tau}\, \mathbf{v} : \mu\alpha.\, \tau} \qquad \frac{\Delta; \Gamma \vdash \mathbf{v} : \tau[\tau'/\alpha] \qquad \Delta \vdash \tau'}{\Delta; \Gamma \vdash \mathbf{pack}\, (\tau', \mathbf{v})\, \mathbf{as}\, \exists \alpha.\, \tau : \exists \alpha.\, \tau}
$$

$\boxed{\Delta; \Gamma \vdash \mathbf{r} : \theta}$

$$
\frac{\Delta; \Gamma \vdash \mathbf{v} : \tau \qquad \Delta \vdash \tau_{exn}}{\Delta; \Gamma \vdash \mathbf{return}\ \mathbf{v} : \mathbf{E}\, \tau_{exn}\, \tau} \qquad \frac{\Delta; \Gamma \vdash \mathbf{v} : \tau_{exn} \qquad \Delta \vdash \tau}{\Delta; \Gamma \vdash \mathbf{raise}\ \mathbf{v} : \mathbf{E}\, \tau_{exn}\, \tau}
$$

$\boxed{\Delta; \Gamma \vdash \mathbf{e} : \theta}$

$$
\frac{\Delta; \Gamma \vdash \mathbf{v} : \tau_1 + \tau_2 \qquad \Delta; \Gamma, \mathbf{x_1} : \tau_1 \vdash \mathbf{e_1} : \theta \qquad \Delta; \Gamma, \mathbf{x_2} : \tau_2 \vdash \mathbf{e_2} : \theta}{\Delta; \Gamma \vdash \mathbf{case}\, \mathbf{v}\, \mathbf{of}\, \mathbf{x_1}.\, \mathbf{e_1} \mid \mathbf{x_2}.\, \mathbf{e_2} : \theta} \qquad \frac{\Delta; \Gamma \vdash \mathbf{v} : \langle \tau_1, \ldots, \tau_n \rangle \qquad \Delta \vdash \tau_{exn}}{\Delta; \Gamma \vdash \mathbf{v.i} : \mathbf{E}\, \tau_{exn}\, \tau_i}
$$

$$
\frac{\Delta; \Gamma \vdash \mathbf{v_1} : \forall[\alpha].\, \tau_2 \to \theta \qquad \Delta \vdash \tau' \qquad \Delta; \Gamma \vdash \mathbf{v_2} : \tau_2[\tau'/\alpha]}{\Delta; \Gamma \vdash \mathbf{v_1}\, [\tau']\, \mathbf{v_2} : \theta[\tau'/\alpha]} \qquad \frac{\Delta; \Gamma \vdash \mathbf{v} : \mu\alpha.\, \tau \qquad \Delta \vdash \tau_{exn}}{\Delta; \Gamma \vdash \mathbf{unfold}\, \mathbf{v} : \mathbf{E}\, \tau_{exn}\, (\tau[\mu\alpha.\, \tau/\alpha])}
$$

$$
\frac{\Delta; \Gamma \vdash \mathbf{v} : \exists \alpha.\, \tau \qquad \Delta, \alpha; \Gamma, \mathbf{x} : \tau \vdash \mathbf{e} : \theta}{\Delta; \Gamma \vdash \mathbf{unpack}\, (\alpha, \mathbf{x}) = \mathbf{v}\, \mathbf{in}\, \mathbf{e} : \theta}
$$

$$
\frac{\Delta; \Gamma \vdash \mathbf{e} : \mathbf{E}\, \tau_{exn}\, \tau \qquad \Delta; \Gamma, \mathbf{x} : \tau \vdash \mathbf{e_1} : \theta \qquad \Delta; \Gamma, \mathbf{y} : \tau_{exn} \vdash \mathbf{e_2} : \theta}{\Delta; \Gamma \vdash \mathbf{handle}\, \mathbf{e}\, \mathbf{with}\, (\mathbf{x}.\, \mathbf{e_1})\, (\mathbf{y}.\, \mathbf{e_2}) : \theta}
$$

Figure 5: Target Language (System F): Static Semantics

$$\mathbf{let}\,\mathbf{x} = \mathbf{e}\,\mathbf{in}\,\mathbf{e'} \stackrel{\text{def}}{=} \mathbf{handle}\,\mathbf{e}\,\mathbf{with}\,(\mathbf{x}.\,\mathbf{e'})\,(\mathbf{y}.\,\mathbf{raise}\ \mathbf{y})$$

$$\mathbf{catch}\ \mathbf{y} = \mathbf{e}\ \mathbf{in}\ \mathbf{e'} \stackrel{\text{def}}{=} \mathbf{handle}\,\mathbf{e}\,\mathbf{with}\,(\mathbf{x}.\,\mathbf{return}\ \mathbf{x})\,(\mathbf{y}.\,\mathbf{e'})$$

$$\mathbf{1} \stackrel{\text{def}}{=} \langle\rangle \qquad \text{(the empty tuple type)}$$

Figure 6: Target Language (System F): Syntax Sugar

3 Closure Conversion

$$
\begin{array}{rcl}
\alpha^+ & = & \boldsymbol{\alpha} \\
\mathbf{1}^+ & = & \mathbf{1} \\
(\sigma_1 + \sigma_2)^+ & = & {\sigma_1}^+ \mathbin{\boldsymbol{+}} {\sigma_2}^+ \\
(\sigma_1 \times \sigma_2)^+ & = & \langle {\sigma_1}^+, {\sigma_2}^+ \rangle \\
(\sigma_1 \to \sigma_2)^+ & = & \exists \boldsymbol{\alpha}.\, \langle (\langle \boldsymbol{\alpha}, {\sigma_1}^+ \rangle \to {\sigma_2}^{\div}), \boldsymbol{\alpha} \rangle \\
(\mu\alpha.\, \sigma)^+ & = & \boldsymbol{\mu\alpha.}\, \sigma^+ \\
\\
\sigma^{\div} & = & \mathbf{E}\,\mathbf{0}\, \sigma^+ \\
\\
(\cdot)^+ & = & \cdot \\
(\Gamma, x : \sigma)^+ & = & \Gamma^+, \mathbf{x} : \sigma^+
\end{array}
$$

Figure 7: Closure Conversion: Type Translation

$$\boxed{\Gamma \vdash \mathsf{v} : \sigma \rightsquigarrow_v \mathbf{v}}$$

$$\frac{\mathsf{x} : \sigma \in \Gamma}{\Gamma \vdash \mathsf{x} : \sigma \rightsquigarrow_v \mathbf{x}} \qquad \frac{}{\Gamma \vdash \langle\rangle : 1 \rightsquigarrow_v \langle\rangle} \qquad \frac{\Gamma \vdash \mathsf{v} : \sigma_1 \rightsquigarrow_v \mathbf{v}}{\Gamma \vdash \mathsf{inj}_1\, \mathsf{v} : \sigma_1 + \sigma_2 \rightsquigarrow_v \mathbf{inj_1\, v}} \qquad \frac{\Gamma \vdash \mathsf{v} : \sigma_2 \rightsquigarrow_v \mathbf{v}}{\Gamma \vdash \mathsf{inj}_2\, \mathsf{v} : \sigma_1 + \sigma_2 \rightsquigarrow_v \mathbf{inj_2\, v}}$$

$$\frac{\Gamma \vdash \mathsf{v}_1 : \sigma_1 \rightsquigarrow_v \mathbf{v_1} \qquad \Gamma \vdash \mathsf{v}_1 : \sigma_1 \rightsquigarrow_v \mathbf{v_1}}{\Gamma \vdash \langle \mathsf{v}_1, \mathsf{v}_2 \rangle : \sigma_1 \times \sigma_2 \rightsquigarrow_v \langle \mathbf{v_1}, \mathbf{v_2} \rangle}$$

$$\frac{\begin{array}{c}\mathrm{fv}(\lambda(\mathsf{x} : \sigma').\, \mathsf{e}) = (\mathsf{y}_1, \ldots, \mathsf{y}_n) \\ \Gamma(\mathsf{y}_i) = \sigma_i \qquad \Gamma' = (\mathsf{y}_1 : \sigma_1, \ldots, \mathsf{y}_n : \sigma_n) \qquad \boldsymbol{\tau_{\mathrm{env}}} = \langle {\sigma_1}^+, \ldots, {\sigma_n}^+ \rangle \qquad \Gamma', \mathsf{x} : \sigma \vdash \mathsf{e} : \sigma' \rightsquigarrow_e \mathbf{e}\end{array}}{\Gamma \vdash \lambda(\mathsf{x} : \sigma).\, \mathsf{e} : \sigma \to \sigma' \rightsquigarrow_v \mathbf{pack}\, (\boldsymbol{\tau_{\mathrm{env}}}, \langle \boldsymbol{\lambda}(\mathbf{z} : \langle \boldsymbol{\tau_{\mathrm{env}}}, \sigma^+ \rangle).\ \begin{array}{l} \mathbf{let\, x_{env} = return_0\, z.1\, in} \\ \mathbf{let\, y_1 = return_0\, x_{env}.1\, in} \\ \vdots \\ \mathbf{let\, y_n = return_0\, x_{env}.n\, in} \\ \mathbf{let\, x = return_0\, z.2\, in\, e} \end{array}, \langle \mathbf{y_1}, \ldots, \mathbf{y_n} \rangle \rangle)\ \mathbf{as}\, \exists \boldsymbol{\alpha}.\, \langle (\langle \boldsymbol{\alpha}, \sigma^+ \rangle \to \mathbf{E\, 0}\, {\sigma'}^+), \boldsymbol{\alpha} \rangle}$$

$$\frac{\Gamma \vdash \mathsf{v} : \sigma[\mu\alpha.\, \sigma / \alpha] \rightsquigarrow_v \mathbf{v}}{\Gamma \vdash \mathsf{fold}_{\mu\alpha.\sigma}\, \mathsf{v} : \mu\alpha.\, \sigma \rightsquigarrow_v \mathbf{fold}_{\boldsymbol{\mu\alpha.\sigma^+}}\, \mathbf{v}}$$

$$\boxed{\Gamma \vdash \mathsf{e} : \sigma \rightsquigarrow_e \mathbf{e}}$$

$$\frac{\Gamma \vdash \mathsf{v} : \sigma \rightsquigarrow_v \mathbf{v}}{\Gamma \vdash \mathsf{v} : \sigma \rightsquigarrow_e \mathbf{return_0\, v}} \qquad \frac{\Gamma \vdash \mathsf{v} : \sigma_1 + \sigma_2 \rightsquigarrow_v \mathbf{v} \qquad \Gamma, \mathsf{x}_1 : \sigma_1 \vdash \mathsf{e}_1 : \sigma \rightsquigarrow_e \mathbf{e_1} \qquad \Gamma, \mathsf{x}_2 : \sigma_2 \vdash \mathsf{e}_2 : \sigma \rightsquigarrow_e \mathbf{e_2}}{\Gamma \vdash \mathsf{case}\, \mathsf{v}\, \mathsf{of}\, \mathsf{x}_1.\, \mathsf{e}_1 \mid \mathsf{x}_2.\, \mathsf{e}_2 : \sigma \rightsquigarrow_e \mathbf{case\, v\, of\, x_1.\, e_1 \mid x_2.\, e_2}}$$

$$\frac{\Gamma \vdash \mathsf{v} : \sigma_i \rightsquigarrow_v \mathbf{v}}{\Gamma \vdash \pi_i\, \mathsf{v} : \sigma_1 \times \sigma_2 \rightsquigarrow_e \mathbf{v.i}} \qquad \frac{\Gamma \vdash \mathsf{v}_1 : \sigma_1 \to \sigma_2 \rightsquigarrow_v \mathbf{v_1} \qquad \Gamma \vdash \mathsf{v}_2 : \sigma_1 \rightsquigarrow_v \mathbf{v_2}}{\Gamma \vdash \mathsf{v}_1\ \mathsf{v}_2 : \sigma_2 \rightsquigarrow_e \begin{array}{l} \mathbf{unpack}\, (\boldsymbol{\alpha}, \mathbf{z}) = \mathbf{v_1\, in} \\ \mathbf{let\, y_1 = return\, z.1\, in} \\ \mathbf{let\, y_2 = return\, z.2\, in} \\ \mathbf{y_1}\ \langle \mathbf{y_2}, \mathbf{v_2} \rangle \end{array}}$$

$$\frac{\Gamma \vdash \mathsf{v} : \mu\alpha.\, \sigma \rightsquigarrow_v \mathbf{v}}{\Gamma \vdash \mathsf{unfold}\, \mathsf{v} : \sigma[\mu\alpha.\, \sigma / \alpha] \rightsquigarrow_e \mathbf{return_0\, unfold\, v}} \qquad \frac{\Gamma \vdash \mathsf{e}_1 : \sigma_1 \rightsquigarrow_e \mathbf{e_1} \qquad \Gamma, \mathsf{x} \vdash \mathsf{e}_2 : \sigma_2 \rightsquigarrow_e \mathbf{e_2}}{\Gamma \vdash \mathsf{let}\, \mathsf{x} = \mathsf{e}_1\, \mathsf{in}\, \mathsf{e}_2 : \sigma_2 \rightsquigarrow_e \mathbf{let\, x = e_1\, in\, e_2}}$$

Figure 8: Closure Conversion: Term Translation

4 Combined Language λ^{ST}

Environments	Γ	$::=$	$\cdot \mid \Gamma, \mathsf{x} : \sigma \mid \Gamma, \mathbf{x} : \boldsymbol{\tau}$
	Δ	$::=$	$\boldsymbol{\Delta}$
Value Types	τ	$::=$	$\sigma \mid \boldsymbol{\tau}$
Computation Types	θ	$::=$	$\sigma \mid \boldsymbol{\theta}$
All Types	φ	$::=$	$\tau \mid \theta$
Variables	x	$::=$	$\mathsf{x} \mid \mathbf{x}$
Values	v	$::=$	$\mathsf{v} \mid \mathbf{v}$
Results	r	$::=$	$\mathsf{v} \mid \mathbf{r}$
Expressions	e	$::=$	$\ldots \mid {}^{\sigma}\mathcal{ST}\,\mathbf{e}$
	$\mathbf{e}$	$::=$	$\ldots \mid \mathcal{TS}^{\sigma}\,\mathsf{e}$
	e	$::=$	$\mathsf{e} \mid \mathbf{e}$
Evaluation Contexts	K	$::=$	$\ldots \mid {}^{\sigma}\mathcal{ST}\,\mathbf{K}$
	$\mathbf{K}$	$::=$	$\ldots \mid \mathcal{TS}^{\sigma}\,\mathsf{K}$
	K	$::=$	$\mathsf{K} \mid \mathbf{K}$

Figure 9: Combined Language (λ^{ST}): Syntax

The syntax of the multi-language is defined by embedding the source and target syntax. Meta-variables defined by ... indicate using the definitions from the corresponding source or target meta-variable. For instance, $\mathbf{p}$ in the multi-language is exactly $\mathbf{p}$ from the target language. However, $\mathbf{e}$ in the multi-language is $\mathbf{e}$ from the target language extended with a boundary term.

Typing in the multi-language, $\Delta; \Gamma \vdash e : \theta$, consists of the typing judgments from both the source and the target languages, with a few modifications. First, the judgments are modified to take the multi-language typing environments Δ and Γ instead of only the source or target typing environments. Next, the typing judgment for the source language is modified at the leaves of each derivation to check that $\Delta \vdash \Gamma$. Finally, two new rules are added to type-check boundary terms, given in Figure 11.

$\boxed{e \overset{\mathsf{ST}}{\hookrightarrow} e'}$

$$
\begin{array}{rcl}
{}^{1}\mathcal{ST}\,\mathbf{return\ v} & \overset{\mathsf{ST}}{\hookrightarrow} & \langle\rangle \\
{}^{\sigma_1+\sigma_2}\mathcal{ST}\,\mathbf{return\ inj_i\ v} & \overset{\mathsf{ST}}{\hookrightarrow} & \mathsf{let}\,\mathsf{x} = {}^{\sigma_i}\mathcal{ST}\,\mathbf{return\ v}\,\mathsf{in}\,\mathsf{inj_i}\,\mathsf{x} \\
{}^{\sigma_1\times\sigma_2}\mathcal{ST}\,\mathbf{return\ v} & \overset{\mathsf{ST}}{\hookrightarrow} & \mathsf{let}\,\mathsf{x_1} = {}^{\sigma_1}\mathcal{ST}\,\mathbf{v.1}\,\mathsf{in}\,\mathsf{let}\,\mathsf{x_2} = {}^{\sigma_2}\mathcal{ST}\,\mathbf{v.2}\,\mathsf{in}\,\langle \mathsf{x_1}, \mathsf{x_2}\rangle \\
{}^{\sigma_1\to\sigma_2}\mathcal{ST}\,\mathbf{return\ v} & \overset{\mathsf{ST}}{\hookrightarrow} & \lambda(\mathsf{x} : \sigma_1).\ {}^{\sigma_2}\mathcal{ST}\left(\begin{array}{l}\mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{z}) = \mathbf{v\ in\ let\ x_f = z.1\ in} \\ \qquad\qquad\qquad\quad \mathbf{let\ x_{env} = z.2\ in} \\ \qquad\qquad\qquad\quad \mathbf{let\ x} = \mathcal{TS}^{\,\sigma_1}\,\mathsf{x}\,\mathbf{in\ x_f}\ [\boldsymbol{\alpha}]\ \langle \mathbf{x_{env}}, \mathbf{x}\rangle\end{array}\right) \\
{}^{\mu\alpha.\sigma}\mathcal{ST}\,\mathbf{return\ v} & \overset{\mathsf{ST}}{\hookrightarrow} & \mathsf{let}\,\mathsf{x} = {}^{\sigma[\mu\alpha.\sigma/\alpha]}\mathcal{ST}\,\mathbf{unfold\ v}\,\mathsf{in}\,\mathsf{fold}_{\mu\alpha.\sigma}\,\mathsf{x} \\
\mathcal{TS}^{\,1}\,\mathsf{v} & \overset{\mathsf{ST}}{\hookrightarrow} & \mathbf{return}\ \langle\rangle \\
\mathcal{TS}^{\,\sigma_1+\sigma_2}\,\mathsf{inj_i}\,\mathsf{v} & \overset{\mathsf{ST}}{\hookrightarrow} & \mathbf{let\ x} = \mathcal{TS}^{\,\sigma_i}\,\mathsf{v}\,\mathbf{in\ return\ inj_i\ x} \\
\mathcal{TS}^{\,\sigma_1\times\sigma_2}\,\mathsf{v} & \overset{\mathsf{ST}}{\hookrightarrow} & \mathbf{let\ x_1} = \mathcal{TS}^{\,\sigma_1}\,\pi_1\,\mathsf{v}\,\mathbf{in\ let\ x_2} = \mathcal{TS}^{\,\sigma_2}\,\pi_2\,\mathsf{v}\,\mathbf{in\ return}\ \langle \mathbf{x_1}, \mathbf{x_2}\rangle \\
\mathcal{TS}^{\,\sigma_1\to\sigma_2}\,\mathsf{v} & \overset{\mathsf{ST}}{\hookrightarrow} & \mathbf{return\ pack}\,(\mathbf{1}, \langle \boldsymbol{\lambda}(\mathbf{z} : \langle \mathbf{1}, {\sigma_1}^{+}\rangle).\ \mathcal{TS}^{\,\sigma_2}\left(\begin{array}{l}\mathsf{let}\,\mathsf{x} = {}^{\sigma_1}\mathcal{ST}\,\mathbf{z.2}\,\mathsf{in} \\ \quad \mathsf{v}\ \mathsf{x}\end{array}\right), \langle\rangle\rangle)\ \mathbf{as}\ (\sigma_1 \to \sigma_2)^{+} \\
\mathcal{TS}^{\,\mu\alpha.\sigma}\,\mathsf{v} & \overset{\mathsf{ST}}{\hookrightarrow} & \mathbf{let\ x} = \mathcal{TS}^{\,\sigma[\mu\alpha.\sigma/\alpha]}\,\mathsf{unfold}\,\mathsf{v}\,\mathbf{in\ return\ fold}_{(\mu\alpha.\sigma)^{+}}\ \mathbf{x}
\end{array}
$$

$$
\frac{\mathsf{e} \overset{\mathsf{S}}{\hookrightarrow} \mathsf{e}'}{K[\mathsf{e}] \longmapsto K[\mathsf{e}']}
\qquad
\frac{\mathbf{e} \overset{\mathbf{T}}{\hookrightarrow} \mathbf{e}'}{K[\mathbf{e}] \longmapsto K[\mathbf{e}']}
\qquad
\frac{e \overset{\mathsf{ST}}{\hookrightarrow} e'}{K[e] \longmapsto K[e']}
$$

Figure 10: Combined Language (λ^{ST}): Operational Semantics

$\boxed{\Delta; \Gamma \vdash e : \theta}$

$$
\frac{\Delta; \Gamma \vdash \mathbf{e} : \sigma^{\div}}{\Delta; \Gamma \vdash {}^{\sigma}\mathcal{ST}\,\mathbf{e} : \sigma}
\qquad\qquad
\frac{\Delta; \Gamma \vdash \mathsf{e} : \sigma}{\Delta; \Gamma \vdash \mathcal{TS}^{\,\sigma}\,\mathsf{e} : \sigma^{\div}}
$$

Figure 11: Combined Language (λ^{ST}): Static Semantics

5 λ^{ST} Contexts and Contextual Equivalence

$$
\begin{array}{lcl}
\mathsf{C^v} & ::= & [\cdot]^{\mathsf{v}} \mid \mathsf{inj_i\ C^v} \mid \langle \mathsf{v}, \mathsf{C^v} \rangle \mid \langle \mathsf{C^v}, \mathsf{v} \rangle \mid \pi_{\mathsf{i}}\ \mathsf{C^v} \mid \lambda(\mathsf{x} : \sigma).\ \mathsf{C} \mid \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{C^v} \\
\mathsf{C} & ::= & [\cdot] \mid [\cdot]^{\mathsf{v}} \mid \mathsf{case\ C^v\ of\ x_1.\,e_1 \mid x_2.\,e_2} \mid \mathsf{case\ v\ of\ x_1.\,C \mid x_2.\,e_2} \\
 & & \mid \mathsf{case\ v\ of\ x_1.\,e_1 \mid x_2.\,C} \mid \mathsf{C^v\ v_2} \mid \mathsf{v_1\ C^v} \mid \mathsf{unfold\ C^v} \mid \mathsf{let\ x = C\ in\ e_2} \mid \mathsf{let\ x = e_1\ in\ C} \mid {}^{\sigma}\mathcal{ST}\ \mathbf{C} \\
\mathbf{C^v} & ::= & [\cdot]^{\mathsf{v}} \mid \mathbf{inj_i\ C^v} \mid \langle \mathbf{v_1}, \ldots, \mathbf{C^v}, \ldots, \mathbf{v_n} \rangle \mid \boldsymbol{\lambda[\alpha](x : \tau).\ C} \mid \mathbf{fold}_{\mu\alpha.\tau}\ \mathbf{C^v} \mid \mathbf{pack}\ (\tau, \mathbf{C^v})\ \mathbf{as}\ \exists\alpha.\,\tau \\
\mathbf{C} & ::= & [\cdot] \mid \mathbf{return\ C^v} \mid \mathbf{raise\ C^v} \mid \mathbf{case\ C^v\ of\ x_1.\,e_1 \mid x_2.\,e_2} \mid \mathbf{case\ v\ of\ x_1.\,C \mid x_2.\,e_2} \mid \mathbf{case\ v\ of\ x_1.\,e_1 \mid x_2.\,C} \\
 & & \mid \mathbf{C^v.i} \mid \mathbf{C^v}\ [\tau]\ \mathbf{v_2} \mid \mathbf{v_1}\ [\tau]\ \mathbf{C^v} \mid \mathbf{unfold\ C^v} \mid \mathbf{unpack}\ (\alpha, \mathbf{x}) = \mathbf{C^v\ in\ e} \mid \mathbf{unpack}\ (\alpha, \mathbf{x}) = \mathbf{v\ in\ C} \\
 & & \mid \mathbf{handle\ C\ with\ (x.\,e_1)\ (y.\,e_2) handle\ e\ with\ (x.\,C)\ (y.\,e_2)} \mid \mathbf{handle\ e\ with\ (x.\,e_1)\ (y.\,C)} \mid \mathcal{TS}^{\,\sigma}\ \mathsf{C} \\
\mathsf{C^g} & ::= & \mathsf{C^v} \mid \mathsf{C} \\
\mathbf{C^g} & ::= & \mathbf{C^v} \mid \mathbf{C} \\
C & ::= & \mathsf{C^g} \mid \mathbf{C^g}
\end{array}
$$

$$\boxed{\vdash C : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \varphi')}$$

$$
\frac{\Delta \vdash \Gamma}{\vdash [\cdot] : (\Delta; \Gamma \vdash \sigma) \Rightarrow (\Delta; \Gamma \vdash \sigma)}
\qquad
\frac{\Delta \vdash \Gamma}{\vdash [\cdot]^{\mathsf{v}} : (\Delta; \Gamma \vdash \sigma) \Rightarrow (\Delta; \Gamma \vdash \sigma)}
\qquad
\frac{\vdash \mathsf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_i)}{\vdash \mathsf{inj_i\ C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_1 + \sigma_2)}
$$

$$
\frac{\vdash \mathsf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_1 + \sigma_2) \qquad \Gamma', \mathsf{x_1} : \sigma_1 \vdash \mathsf{e_1} : \sigma \qquad \Gamma', \mathsf{x_2} : \sigma_2 \vdash \mathsf{e_2} : \sigma}{\vdash \mathsf{case\ C^v\ of\ x_1.\,e_1 \mid x_2.\,e_2} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma)}
$$

$$
\frac{\vdash \mathsf{C} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma', \mathsf{x_1} : \sigma_1 \vdash \sigma) \qquad \Gamma' \vdash \mathsf{v} : \sigma_1 + \sigma_2 \qquad \Gamma', \mathsf{x_2} : \sigma_2 \vdash \mathsf{e_2} : \sigma}{\vdash \mathsf{case\ v\ of\ x_1.\,C \mid x_2.\,e_2} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma)}
$$

$$
\frac{\vdash \mathsf{C} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma', \mathsf{x_2} : \sigma_2 \vdash \sigma) \qquad \Gamma' \vdash \mathsf{v} : \sigma_1 + \sigma_2 \qquad \Gamma', \mathsf{x_1} : \sigma_1 \vdash \mathsf{e_1} : \sigma}{\vdash \mathsf{case\ v\ of\ x_1.\,e_1 \mid x_2.\,C} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma)}
$$

$$
\frac{\Delta' \vdash \Gamma' : \mathsf{v}\sigma_1 \qquad \vdash \mathsf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_2)}{\vdash \langle \mathsf{v}, \mathsf{C^v} \rangle : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_1 \times \sigma_2)}
$$

$$
\frac{\vdash \mathsf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_1) \qquad \Delta' \vdash \Gamma' : \mathsf{v}\sigma_2}{\vdash \langle \mathsf{C^v}, \mathsf{v} \rangle : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_1 \times \sigma_2)}
\quad
\frac{\vdash \mathsf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_1 \times \sigma_2)}{\vdash \pi_{\mathsf{i}}\ \mathsf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_i)}
$$

$$
\frac{\vdash \mathsf{C} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma', \mathsf{x} : \sigma_1 \vdash \sigma_2)}{\vdash \lambda(\mathsf{x} : \sigma_1).\ \mathsf{C} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_1 \to \sigma_2)}
\qquad
\frac{\vdash \mathsf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_1 \to \sigma_2) \qquad \Delta'; \Gamma' \vdash \mathsf{v_2} : \sigma_1}{\vdash \mathsf{C^v\ v_2} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_2)}
$$

$$
\frac{\Delta'; \Gamma' \vdash \mathsf{v_1} : \sigma_1 \to \sigma_2 \qquad \vdash \mathsf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_1)}{\vdash \mathsf{v_1\ C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_2)}
\qquad
\frac{\vdash \mathsf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma'[\mu\alpha.\,\sigma'/\alpha])}{\vdash \mathsf{fold}_{\mu\alpha.\sigma'}\ \mathsf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \mu\alpha.\,\sigma')}
$$

$$
\frac{\vdash \mathsf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \mu\alpha.\,\sigma')}{\vdash \mathsf{unfold\ C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma'[\mu\alpha.\,\sigma'/\alpha])}
\qquad
\frac{\vdash \mathsf{C} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_1) \qquad \Delta'; \Gamma', \mathsf{x} : \sigma_1 \vdash \mathsf{e_2} : \sigma_2}{\vdash \mathsf{let\ x = C\ in\ e_2} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_2)}
$$

$$
\frac{\Delta'; \Gamma' \vdash \mathsf{e_1} : \sigma_1 \qquad \vdash \mathsf{C} : (\Delta; \Gamma, \mathsf{x} : \sigma_1 \vdash \varphi) \Rightarrow (\Delta'; \Gamma', \mathsf{x} : \sigma_1 \vdash \sigma_2)}{\vdash \mathsf{let\ x = e_1\ in\ C} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma_2)}
\qquad
\frac{\vdash \mathbf{C} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma^{\div})}{\vdash {}^{\sigma}\mathcal{ST}\ \mathbf{C} : (\Delta'; \Gamma' \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \sigma)}
$$

Figure 12: λ^{ST} Contexts and Context Typing

$$\frac{\Delta \vdash \Gamma}{\vdash [\cdot]^{\mathbf{v}} : (\Delta; \Gamma \vdash \boldsymbol{\tau}) \Rightarrow (\Delta; \Gamma \vdash \boldsymbol{\tau})} \qquad \frac{\Delta \vdash \Gamma}{\vdash [\cdot] : (\Delta; \Gamma \vdash \boldsymbol{\theta}) \Rightarrow (\Delta; \Gamma \vdash \boldsymbol{\theta})}$$

$$\frac{\vdash \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\tau})}{\vdash \mathbf{return}_{\boldsymbol{\tau}_{\mathbf{exn}}}\ \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \mathbf{E}\,\boldsymbol{\tau}\,\boldsymbol{\tau}_{\mathbf{exn}})} \qquad \frac{\vdash \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\tau}_{\mathbf{exn}})}{\vdash \mathbf{raise}_{\boldsymbol{\tau}}\ \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \mathbf{E}\,\boldsymbol{\tau}\,\boldsymbol{\tau}_{\mathbf{exn}})}$$

$$\frac{\vdash \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\tau}_{\mathbf{i}})}{\vdash \mathbf{inj_i}\ \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\tau}_{\mathbf{1}} + \boldsymbol{\tau}_{\mathbf{2}})}$$

$$\frac{\Delta'; \Gamma' \vdash \mathbf{v_i} : \boldsymbol{\tau}_{\mathbf{i}} \qquad \vdash \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\tau})}{\vdash \langle \mathbf{v_1}, \ldots, \mathbf{C^v}, \ldots, \mathbf{v_n} \rangle : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \langle \boldsymbol{\tau}_{\mathbf{1}}, \ldots, \boldsymbol{\tau}, \ldots, \boldsymbol{\tau}_{\mathbf{n}} \rangle)}$$

$$\frac{\vdash \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \langle \boldsymbol{\tau}_{\mathbf{1}}, \ldots, \boldsymbol{\tau}_{\mathbf{i}}, \ldots, \boldsymbol{\tau}_{\mathbf{n}} \rangle) \qquad \Delta' \vdash \boldsymbol{\tau}_{\mathbf{exn}}}{\vdash \mathbf{C^v.i} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \mathbf{E}\,\boldsymbol{\tau}_{\mathbf{i}}\,\boldsymbol{\tau}_{\mathbf{exn}})}$$

$$\frac{\vdash \mathbf{C} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\boldsymbol{\alpha}; \mathbf{x} : \boldsymbol{\tau} \vdash \boldsymbol{\theta})}{\vdash \boldsymbol{\lambda}[\boldsymbol{\alpha}](\mathbf{x} : \boldsymbol{\tau}).\, \mathbf{C} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \forall[\boldsymbol{\alpha}].\, \boldsymbol{\tau} \to \boldsymbol{\theta})}$$

$$\frac{\vdash \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \forall[\boldsymbol{\alpha}].\, \boldsymbol{\tau} \to \boldsymbol{\theta}) \qquad \Delta'; \Gamma' \vdash \mathbf{v_2} : \boldsymbol{\tau}[\boldsymbol{\tau}'/\boldsymbol{\alpha}] \qquad \Delta' \vdash \boldsymbol{\tau}'}{\vdash \mathbf{C^v}\ [\boldsymbol{\tau}']\ \mathbf{v_2} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\theta}[\boldsymbol{\tau}'/\boldsymbol{\alpha}])}$$

$$\frac{\Delta'; \Gamma' \vdash \mathbf{v_1} : \forall[\boldsymbol{\alpha}].\, \boldsymbol{\tau} \to \boldsymbol{\theta} \qquad \vdash \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\tau}[\boldsymbol{\tau}'/\boldsymbol{\alpha}]) \qquad \Delta' \vdash \boldsymbol{\tau}'}{\vdash \mathbf{v_1}\ [\boldsymbol{\tau}']\ \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\theta}[\boldsymbol{\tau}'/\boldsymbol{\alpha}])}$$

$$\frac{\vdash \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\tau}[\boldsymbol{\mu\alpha}.\, \boldsymbol{\tau}/\boldsymbol{\alpha}])}{\vdash \mathbf{fold}_{\boldsymbol{\mu\alpha}.\boldsymbol{\tau}}\ \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\mu\alpha}.\, \boldsymbol{\tau})} \qquad \frac{\vdash \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\mu\alpha}.\, \boldsymbol{\tau}) \qquad \Delta' \vdash \boldsymbol{\tau}_{\mathbf{exn}}}{\vdash \mathbf{unfold}\ \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \mathbf{E}\,\boldsymbol{\tau}_{\mathbf{exn}}\,\boldsymbol{\tau}[\boldsymbol{\mu\alpha}.\, \boldsymbol{\tau}/\boldsymbol{\alpha}])}$$

$$\frac{\vdash \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\tau}_{\mathbf{1}}[\boldsymbol{\tau}_{\mathbf{2}}/\boldsymbol{\alpha}])}{\vdash \mathbf{pack}\,(\boldsymbol{\tau}_{\mathbf{2}}, \mathbf{C^v})\ \mathbf{as}\ \exists \boldsymbol{\alpha}.\, \boldsymbol{\tau}_{\mathbf{1}} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \exists \boldsymbol{\alpha}.\, \boldsymbol{\tau}_{\mathbf{1}})}$$

$$\frac{\vdash \mathbf{C^v} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \exists \boldsymbol{\alpha}.\, \boldsymbol{\tau}) \qquad \Delta', \boldsymbol{\alpha}; \Gamma', \mathbf{x} : \boldsymbol{\tau} \vdash \mathbf{e} : \boldsymbol{\theta}}{\vdash \mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{x}) = \mathbf{C^v}\ \mathbf{in}\ \mathbf{e} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\theta})}$$

$$\frac{\Delta'; \Gamma' \vdash \mathbf{v} : \exists \boldsymbol{\alpha}.\, \boldsymbol{\tau} \qquad \vdash \mathbf{C} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta', \boldsymbol{\alpha}; \Gamma', \mathbf{x} : \boldsymbol{\tau} \vdash \boldsymbol{\theta})}{\vdash \mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{x}) = \mathbf{v}\ \mathbf{in}\ \mathbf{C} : (\Delta; \Gamma \vdash \varphi) \Rightarrow (\Delta'; \Gamma' \vdash \boldsymbol{\theta})}$$

Figure 13: λ^{ST} Context Typing (continued)

$$\frac{\vdash \mathbf{C^v} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma' \vdash \boldsymbol{\tau_1 + \tau_2}) \qquad \Delta';\Gamma', \mathbf{x_1} : \boldsymbol{\tau_1} \vdash \mathbf{e_1} : \boldsymbol{\theta} \qquad \Delta';\Gamma', \mathbf{x_2} : \boldsymbol{\tau_2} \vdash \mathbf{e_2} : \boldsymbol{\theta}}{\vdash \mathbf{case\, C^v\, of\, x_1.\, e_1 \mid x_2.\, e_2} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma' \vdash \boldsymbol{\theta})}$$

$$\frac{\vdash \mathbf{C} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma', \mathbf{x_1} : \boldsymbol{\tau_1} \vdash \boldsymbol{\theta}) \qquad \Delta';\Gamma' \vdash \mathbf{v} : \boldsymbol{\tau_1 + \tau_2} \qquad \Delta';\Gamma', \mathbf{x_2} : \boldsymbol{\tau_2} \vdash \mathbf{e_2} : \boldsymbol{\theta}}{\vdash \mathbf{case\, v\, of\, x_1.\, C \mid x_2.\, e_2} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma' \vdash \boldsymbol{\theta})}$$

$$\frac{\vdash \mathbf{C} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma', \mathbf{x_2} : \boldsymbol{\tau_2} \vdash \boldsymbol{\theta}) \qquad \Delta';\Gamma' \vdash \mathbf{v} : \boldsymbol{\tau_1 + \tau_2} \qquad \Delta';\Gamma', \mathbf{x_1} : \boldsymbol{\tau_1} \vdash \mathbf{e_1} : \boldsymbol{\theta}}{\vdash \mathbf{case\, v\, of\, x_1.\, e_1 \mid x_2.\, C} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma' \vdash \boldsymbol{\theta})}$$

$$\frac{\vdash \mathbf{C} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma' \vdash \mathbf{E}\, \boldsymbol{\tau'_{\mathrm{exn}}}\, \boldsymbol{\tau'}) \qquad \Delta';\Gamma', x : \boldsymbol{\tau'} \vdash \mathbf{e_1} : \mathbf{E}\, \boldsymbol{\tau_{\mathrm{exn}}}\, \boldsymbol{\tau} \qquad \Delta';\Gamma', y : \boldsymbol{\tau'_{\mathrm{exn}}} \vdash \mathbf{e_2} : \mathbf{E}\, \boldsymbol{\tau_{\mathrm{exn}}}\, \boldsymbol{\tau}}{\vdash \mathbf{handle\, C\, with\, (x.\, e_1)\, (y.\, e_2)} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma' \vdash \mathbf{E}\, \boldsymbol{\tau_{\mathrm{exn}}}\, \boldsymbol{\tau})}$$

$$\frac{\Delta';\Gamma' \vdash \mathbf{e} : \mathbf{E}\, \boldsymbol{\tau'_{\mathrm{exn}}}\, \boldsymbol{\tau'} \qquad \vdash \mathbf{C} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma', \mathbf{x} : \boldsymbol{\tau'} \vdash \mathbf{E}\, \boldsymbol{\tau_{\mathrm{exn}}}\, \boldsymbol{\tau}) \qquad \Delta';\Gamma', \mathbf{y} : \boldsymbol{\tau'_{\mathrm{exn}}} \vdash \mathbf{e_2} : \mathbf{E}\, \boldsymbol{\tau_{\mathrm{exn}}}\, \boldsymbol{\tau}}{\vdash \mathbf{handle\, e\, with\, (x.\, C)\, (y.\, e_2)} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma' \vdash \mathbf{E}\, \boldsymbol{\tau_{\mathrm{exn}}}\, \boldsymbol{\tau})}$$

$$\frac{\Delta';\Gamma' \vdash \mathbf{e} : \mathbf{E}\, \boldsymbol{\tau'_{\mathrm{exn}}}\, \boldsymbol{\tau'} \qquad \Delta';\Gamma', \mathbf{x} : \boldsymbol{\tau'} \vdash \mathbf{e_1} : \mathbf{E}\, \boldsymbol{\tau_{\mathrm{exn}}}\, \boldsymbol{\tau} \qquad \vdash \mathbf{C} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma', \mathbf{y} : \boldsymbol{\tau'_{\mathrm{exn}}} \vdash \mathbf{E}\, \boldsymbol{\tau_{\mathrm{exn}}}\, \boldsymbol{\tau})}{\vdash \mathbf{handle\, e\, with\, (x.\, e_1)\, (y.\, C)} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma' \vdash \mathbf{E}\, \boldsymbol{\tau_{\mathrm{exn}}}\, \boldsymbol{\tau})}$$

$$\frac{\vdash \mathsf{C} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma' \vdash \sigma')}{\vdash \mathcal{TS}^{\sigma'}\, \mathsf{C} : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma' \vdash (\sigma')^{+})}$$

Figure 14: λ^{ST} Context Typing (continued, continued)

$$\begin{aligned}
\Delta \vDash \delta \quad &\overset{\mathrm{def}}{=} \quad \forall \boldsymbol{\alpha} \in \Delta.\ \Delta \vdash \delta(\boldsymbol{\alpha}) \\
\delta, \Gamma \vDash \gamma \quad &\overset{\mathrm{def}}{=} \quad \forall x : \tau \in \Gamma.\ \cdot;\cdot \vdash \gamma(x) : \delta(\tau) \\
\Delta;\Gamma \vdash e_1 \approx^{ciu}_{\mathrm{ST}} e_2 : \theta \quad &\overset{\mathrm{def}}{=} \quad \Delta;\Gamma \vdash e_1 : \theta \ \wedge\ \Delta;\Gamma \vdash e_2 : \theta \ \wedge \\
&\qquad \forall \delta, \gamma, K.\ (\Delta \vDash \delta \wedge \delta, \Gamma \vDash \gamma \wedge \vdash K : (\cdot;\cdot \vdash \theta) \Rightarrow (\cdot;\cdot \vdash \mathsf{1})) \implies \\
&\qquad (K[\delta(\gamma(e_1))] \Updownarrow K[\delta(\gamma(e_2))])
\end{aligned}$$

Figure 15: CIU Equivalence

$$\begin{aligned}
\mathsf{\Gamma} \vdash \mathsf{e_1} \approx^{ctx}_{\mathsf{S}} \mathsf{e_2} : \sigma \quad &\overset{\mathrm{def}}{=} \quad \mathsf{\Gamma} \vdash \mathsf{e_1} : \sigma \ \wedge\ \mathsf{\Gamma} \vdash \mathsf{e_2} : \sigma \ \wedge \\
&\qquad \forall \mathsf{C}.\ \text{source } \mathsf{C} \ \wedge\ \vdash \mathsf{C} : (\cdot;\mathsf{\Gamma} \vdash \sigma) \Rightarrow (\cdot;\cdot \vdash \mathsf{1}) \\
&\qquad\quad \implies (\mathsf{C}[\mathsf{e_1}] \Updownarrow \mathsf{C}[\mathsf{e_2}]) \\
\boldsymbol{\Delta};\boldsymbol{\Gamma} \vdash \mathbf{e_1} \approx^{ctx}_{\mathbf{T}} \mathbf{e_2} : \boldsymbol{\theta} \quad &\overset{\mathrm{def}}{=} \quad \boldsymbol{\Delta};\boldsymbol{\Gamma} \vdash \mathbf{e_1} : \boldsymbol{\theta} \ \wedge\ \boldsymbol{\Delta};\boldsymbol{\Gamma} \vdash \mathbf{e_2} : \boldsymbol{\theta} \ \wedge \\
&\qquad \forall \mathbf{C}.\ \text{target } \mathbf{C} \ \wedge\ \vdash \mathbf{C} : (\boldsymbol{\Delta};\boldsymbol{\Gamma} \vdash \boldsymbol{\theta}) \Rightarrow (\cdot;\cdot \vdash \mathbf{E\,0\,1}) \\
&\qquad\quad \implies (\mathbf{C}[\mathbf{e_1}] \Updownarrow \mathbf{C}[\mathbf{e_2}]) \\
\Delta;\Gamma \vdash e_1 \approx^{ctx}_{\mathrm{ST}} e_2 : \theta \quad &\overset{\mathrm{def}}{=} \quad \Delta;\Gamma \vdash e_1 : \theta \ \wedge\ \Delta;\Gamma \vdash e_2 : \theta \ \wedge \\
&\qquad \forall C.\ \vdash C : (\Delta;\Gamma \vdash \theta) \Rightarrow (\cdot;\cdot \vdash \mathsf{1}) \\
&\qquad\quad \implies (C[e_1] \Updownarrow C[e_2])
\end{aligned}$$

Figure 16: Source, Target and Multi-language Contextual Equivalence

6 λ^{ST} Logical Relation

$$
\begin{array}{rcl}
\text{running}(k, e) & \stackrel{\text{def}}{=} & \exists e'.\ e \longmapsto^{k} e' \\
\text{Atom}[\varphi_1, \varphi_2] & \stackrel{\text{def}}{=} & \{\, (k, e_1, e_2) \mid\ k \in \mathbb{N} \ \wedge\ \cdot ; \cdot \vdash e_1 : \varphi_1 \ \wedge\ \cdot ; \cdot \vdash e_2 : \varphi_2 \,\} \\
\text{Atom}[\varphi]\rho & \stackrel{\text{def}}{=} & \text{Atom}[\rho_1(\varphi), \rho_2(\varphi)] \\
\text{Atom}^{\text{val}}[\tau_1, \tau_2] & \stackrel{\text{def}}{=} & \{\, (k, v_1, v_2) \mid (k, v_1, v_2) \in \text{Atom}[\tau_1, \tau_2] \,\} \\
\text{Atom}^{\text{val}}[\tau]\rho & \stackrel{\text{def}}{=} & \text{Atom}^{\text{val}}[\rho_1(\tau), \rho_2(\tau)] \\
\text{Atom}^{\text{res}}[\theta_1, \theta_2] & \stackrel{\text{def}}{=} & \{\, (k, r_1, r_2) \mid (k, r_1, r_2) \in \text{Atom}[\theta_1, \theta_2] \,\} \\
\text{Atom}^{\text{res}}[\theta]\rho & \stackrel{\text{def}}{=} & \text{Atom}^{\text{res}}[\rho_1(\theta), \rho_2(\theta)] \\
\text{Atom}^{\mathcal{K}}[\theta_1, \theta_2] & \stackrel{\text{def}}{=} & \{\, (k, K_1, K_2) \mid\ k \in \mathbb{N} \ \wedge\ \exists \theta_1', \theta_2'.\ \vdash K_1 : (\cdot ; \cdot \vdash \theta_1) \Rightarrow (\cdot ; \cdot \vdash \theta_1') \ \wedge\ \vdash K_2 : (\cdot ; \cdot \vdash \theta_2) \Rightarrow (\cdot ; \cdot \vdash \theta_2') \} \\
\text{Atom}^{\mathcal{K}}[\theta]\rho & \stackrel{\text{def}}{=} & \text{Atom}^{\mathcal{K}}[\rho_1(\theta), \rho_2(\theta)] \\
\text{Rel}[\boldsymbol{\tau_1}, \boldsymbol{\tau_2}] & \stackrel{\text{def}}{=} & \{\, R \in \mathscr{P}(\text{Atom}^{\text{val}}[\boldsymbol{\tau_1}, \boldsymbol{\tau_2}]) \mid \forall (k, \mathbf{v_1}, \mathbf{v_2}) \in R.\ \ \forall j < k.\ \ (j, \mathbf{v_1}, \mathbf{v_2}) \in R \,\}
\end{array}
$$

Figure 17: Logical Relation Auxiliary Definitions

$$
\begin{array}{rcl}
\mathcal{V}\,[\![\tau]\!]\,\rho & \subset & \mathrm{Atom}^{\mathrm{val}}[\tau]\rho \\
\mathcal{V}\,[\![\mathsf{1}]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, \langle\rangle, \langle\rangle)\,\} \\
\mathcal{V}\,[\![\sigma_1 + \sigma_2]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, \mathsf{inj_i}\,\mathsf{v}_1, \mathsf{inj_i}\,\mathsf{v}_2) \mid \mathsf{i} \in \{\mathsf{1}, \mathsf{2}\} \;\wedge\; (k, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}\,[\![\sigma_i]\!]\,\rho\,\} \\
\mathcal{V}\,[\![\sigma \times \sigma']\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, \langle \mathsf{v}_1, \mathsf{v}_1'\rangle, \langle \mathsf{v}_2, \mathsf{v}_2'\rangle) \mid (k, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}\,[\![\sigma]\!]\,\rho \;\wedge\; (k, \mathsf{v}_1', \mathsf{v}_2') \in \mathcal{V}\,[\![\sigma']\!]\,\rho\,\} \\
\mathcal{V}\,[\![\sigma \to \sigma']\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, \lambda(\mathsf{x}:\sigma).\,\mathsf{e}_1, \lambda(\mathsf{x}:\sigma).\,\mathsf{e}_2) \mid \forall j \le k.\ \forall \mathsf{v}_1, \mathsf{v}_2.\ (j, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}\,[\![\sigma]\!]\,\rho \implies (j, \mathsf{e}_1[\mathsf{v}_1/\mathsf{x}], \mathsf{e}_2[\mathsf{v}_2/\mathsf{x}]) \in \mathcal{E}\,[\![\sigma']\!]\,\rho\,\} \\
\mathcal{V}\,[\![\mu\alpha.\,\sigma]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, \mathsf{fold}_{\mu\alpha.\sigma}\,\mathsf{v}_1, \mathsf{fold}_{\mu\alpha.\sigma}\,\mathsf{v}_2) \mid \forall j < k.\ \ (j, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}\,[\![\sigma[\mu\alpha.\,\sigma/\alpha]]\!]\,\rho\,\} \\
\mathcal{V}\,[\![\boldsymbol{\alpha}]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \rho_R(\boldsymbol{\alpha}) \\
\mathcal{V}\,[\![\boldsymbol{\tau_1} + \boldsymbol{\tau_2}]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, \mathbf{inj_i}\,\mathbf{v_1}, \mathbf{inj_i}\,\mathbf{v_2}) \mid \mathbf{i} \in \{\mathbf{1}, \mathbf{2}\} \;\wedge\; (k, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}\,[\![\boldsymbol{\tau}_\mathbf{i}]\!]\,\rho\} \\
\mathcal{V}\,[\![\langle \boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_n}\rangle]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, \langle \mathbf{v_1}, \ldots, \mathbf{v_n}\rangle, \langle \mathbf{v_1'}, \ldots, \mathbf{v_n'}\rangle) \mid \forall i \in \{1 \ldots n\}.\ (k, \mathbf{v_i}, \mathbf{v_i'}) \in \mathcal{V}\,[\![\boldsymbol{\tau}_\mathbf{i}]\!]\,\rho\,\} \\
\mathcal{V}\,[\![\forall[\boldsymbol{\alpha}].\,\boldsymbol{\tau} \to \boldsymbol{\theta}]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, \boldsymbol{\lambda}[\boldsymbol{\alpha}](\mathbf{x} : \rho_1(\boldsymbol{\tau})).\,\mathbf{e_1}, \boldsymbol{\lambda}[\boldsymbol{\alpha}](\mathbf{x} : \rho_2(\boldsymbol{\tau})).\,\mathbf{e_2}) \mid \\
 & & \quad \forall \boldsymbol{\tau_1}, \boldsymbol{\tau_2}, R \in \mathrm{Rel}[\boldsymbol{\tau_1}, \boldsymbol{\tau_2}]. \\
 & & \quad \forall j \le k.\ \forall (j, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}\,[\![\boldsymbol{\tau}]\!]\,\rho[\boldsymbol{\alpha} \mapsto (\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, R)]. \\
 & & \qquad\quad (j, \mathbf{e_1}[\boldsymbol{\tau_1}/\boldsymbol{\alpha}][\mathbf{v_1}/\mathbf{x}], \mathbf{e_2}[\boldsymbol{\tau_2}/\boldsymbol{\alpha}][\mathbf{v_2}/\mathbf{x}]) \in \mathcal{E}\,[\![\boldsymbol{\theta}]\!]\,\rho[\boldsymbol{\alpha} \mapsto (\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, R)]\,\} \\
\mathcal{V}\,[\![\boldsymbol{\mu\alpha}.\,\boldsymbol{\tau}]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, \mathbf{fold}_{\rho_1(\boldsymbol{\mu\alpha}.\boldsymbol{\tau})}\,\mathbf{v_1}, \mathbf{fold}_{\rho_2(\boldsymbol{\mu\alpha}.\boldsymbol{\tau})}\,\mathbf{v_2}) \mid \forall j < k.\ (j, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}\,[\![\boldsymbol{\tau}[\boldsymbol{\mu\alpha}.\,\boldsymbol{\tau}/\boldsymbol{\alpha}]]\!]\,\rho\,\} \\
\mathcal{V}\,[\![\mathbf{0}]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \emptyset \\
\mathcal{V}\,[\![\exists\boldsymbol{\alpha}.\,\boldsymbol{\tau}]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, \mathbf{pack}\,(\boldsymbol{\tau_1}, \mathbf{v_1})\,\mathbf{as}\,\rho_1(\exists\boldsymbol{\alpha}.\,\boldsymbol{\tau}), \mathbf{pack}\,(\boldsymbol{\tau_2}, \mathbf{v_2})\,\mathbf{as}\,\rho_2(\exists\boldsymbol{\alpha}.\,\boldsymbol{\tau})) \mid \\
 & & \quad \exists R \in \mathrm{Rel}[\boldsymbol{\tau_1}, \boldsymbol{\tau_2}].\ (k, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}\,[\![\boldsymbol{\tau}]\!]\,\rho[\boldsymbol{\alpha} \mapsto (\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, R)]\,\} \\
\mathcal{R}[\![\theta]\!]\rho & \subset & \mathrm{Atom}^{\mathrm{res}}[\theta]\rho \\
\mathcal{R}[\![\sigma]\!]\rho & \stackrel{\mathrm{def}}{=} & \mathcal{V}\,[\![\sigma]\!]\,\rho \\
\mathcal{R}[\![\mathbf{E}\,\boldsymbol{\tau}_{\mathbf{exn}}\,\boldsymbol{\tau}]\!]\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, \mathbf{return}\ \mathbf{v_1}, \mathbf{return}\ \mathbf{v_2}) \mid (k, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}\,[\![\boldsymbol{\tau}]\!]\,\rho\,\} \\
 & & \cup \\
 & & \{\,(k, \mathbf{raise}\ \mathbf{v_1}, \mathbf{raise}\ \mathbf{v_2}) \mid (k, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}\,[\![\boldsymbol{\tau}_{\mathbf{exn}}]\!]\,\rho\,\} \\
\mathcal{E}\,[\![\theta]\!]\,\rho & \subset & \mathrm{Atom}[\theta]\rho \\
\mathcal{E}\,[\![\theta]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, e_1, e_2) \mid \forall K_1, K_2.\ (k, K_1, K_2) \in \mathcal{K}\,[\![\theta]\!]\,\rho \implies (k, K_1[e_1], K_2[e_2]) \in \mathcal{O}\} \\
\mathcal{K}\,[\![\theta]\!]\,\rho & \subset & \mathrm{Atom}^{\mathcal{K}}[\theta]\rho \\
\mathcal{K}\,[\![\theta]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, K_1, K_2) \mid\ \forall j \le k, r_1, r_2.\ (j, r_1, r_2) \in \mathcal{R}[\![\theta]\!]\rho \implies (j, K_1[r_1], K_2[r_2]) \in \mathcal{O}\} \\
\mathcal{O} & \stackrel{\mathrm{def}}{=} & \{\,(k, e_1, e_2) \mid\ (e_1 \Downarrow\ \wedge\ e_2 \Downarrow) \;\vee\; (\mathrm{running}(k, e_1) \;\wedge\; \mathrm{running}(k, e_2))\,\} \\
\mathcal{D}\,[\![\cdot]\!] & \stackrel{\mathrm{def}}{=} & \{\,\emptyset\,\} \\
\mathcal{D}\,[\![\Delta, \boldsymbol{\alpha}]\!] & \stackrel{\mathrm{def}}{=} & \{\,\rho[\boldsymbol{\alpha} \mapsto (\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, R)] \mid\ \rho \in \mathcal{D}\,[\![\Delta]\!]\ \;\wedge\; R \in \mathrm{Rel}[\boldsymbol{\tau_1}, \boldsymbol{\tau_2}]\,\} \\
\mathcal{G}\,[\![\cdot]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, \emptyset) \mid\ k \in \mathbb{N}\,\} \\
\mathcal{G}\,[\![\Gamma, x : \tau]\!]\,\rho & \stackrel{\mathrm{def}}{=} & \{\,(k, \gamma[x \mapsto (v_1, v_2)]) \mid\ (k, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho \;\wedge\; (k, v_1, v_2) \in \mathcal{V}\,[\![\tau]\!]\,\rho\,\}
\end{array}
$$

Figure 18: Combined Language (λ^{ST}): Logical Relations for Closed Terms

$$
\begin{aligned}
\Delta;\Gamma \vdash e_1 \approx^{log}_{\mathcal{E}} e_2 : \theta \quad &\stackrel{\text{def}}{=} \quad \Delta;\Gamma \vdash e_1 : \theta \;\wedge\; \Delta;\Gamma \vdash e_2 : \theta \;\wedge \\
&\qquad \forall k \geq 0.\ \forall \rho, \gamma.\ \rho \in \mathcal{D}\,[\![\Delta]\!] \;\wedge\; (k,\gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho \implies \\
&\qquad (k, \rho_1(\gamma_1(e_1)), \rho_2(\gamma_2(e_2))) \in \mathcal{E}\,[\![\theta]\!]\,\rho \\
\Delta;\Gamma \vdash v_1 \approx^{log}_{\mathcal{V}} v_2 : \tau \quad &\stackrel{\text{def}}{=} \quad \Delta;\Gamma \vdash v_1 : \tau \;\wedge\; \Delta;\Gamma \vdash v_2 : \tau \;\wedge \\
&\qquad \forall k \geq 0.\ \forall \rho, \gamma.\ \rho \in \mathcal{D}\,[\![\Delta]\!] \;\wedge\; (k,\gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho \implies \\
&\qquad (k, \rho_1(\gamma_1(v_1)), \rho_2(\gamma_2(v_2))) \in \mathcal{V}\,[\![\tau]\!]\,\rho \\
\vdash C_1 \approx^{log}_{\mathcal{I}\Rightarrow\mathcal{J}} C_2 : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma' \vdash \varphi') \quad &\stackrel{\text{def}}{=} \quad \vdash C_1 : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma' \vdash \varphi') \;\wedge \\
&\qquad \vdash C_2 : (\Delta;\Gamma \vdash \varphi) \Rightarrow (\Delta';\Gamma' \vdash \varphi') \\
&\qquad \wedge\; \forall e_1, e_2.\ \Delta;\Gamma \vdash e_1 \approx^{log}_{\mathcal{I}} e_2 : \varphi \implies \\
&\qquad \Delta';\Gamma' \vdash C_1[e_1] \approx^{log}_{\mathcal{J}} C_2[e_2] : \varphi'
\end{aligned}
$$

Figure 19: Combined Language (λ^{ST}): Logical Relations for Open Terms

$$
\begin{aligned}
\mathcal{V}^{+}[\![\sigma]\!] \quad &\stackrel{\text{def}}{=} \quad \{(k, \mathsf{v_1}, \mathbf{v_2}) \in \mathrm{Atom}[\sigma, \sigma^{+}] \mid \\
&\qquad \exists \mathsf{v_2}.\ {}^{\sigma}\mathcal{ST}\,\mathbf{v_2} \longmapsto^{*} \mathsf{v_2} \wedge (k, \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}\,[\![\sigma]\!]\,\emptyset\} \\
\mathcal{E}^{\div}[\![\sigma]\!] \quad &\stackrel{\text{def}}{=} \quad \{(k, \mathsf{e}, \mathbf{e}) \in \mathrm{Atom}[\sigma, \sigma^{\div}] \mid (k, \mathsf{e}, {}^{\sigma}\mathcal{ST}\,\mathbf{e}) \in \mathcal{E}\,[\![\sigma]\!]\,\emptyset\} \\
\mathcal{G}^{+}[\![\cdot]\!] \quad &\stackrel{\text{def}}{=} \quad \{(k, \emptyset, \emptyset) \mid k \in \mathbb{N}\} \\
\mathcal{G}^{+}[\![\Gamma, \mathsf{x} : \sigma]\!] \quad &\stackrel{\text{def}}{=} \quad \{(k, \gamma[\mathsf{x} \mapsto \mathsf{v}], \boldsymbol{\gamma}[\mathbf{x} \mapsto \mathbf{v}]) \mid \\
&\qquad (k, \gamma, \boldsymbol{\gamma}) \in \mathcal{G}^{+}[\![\Gamma]\!] \wedge (k, \mathsf{v}, \mathbf{v}) \in \mathcal{V}^{+}[\![\sigma]\!]\} \\
\Gamma \vdash \mathsf{v} \approx_{+} \mathbf{v} : \sigma \quad &\stackrel{\text{def}}{=} \quad \mathsf{v} \in \lambda^{\mathrm{S}} \wedge \mathbf{v} \in \lambda^{\mathrm{T}} \wedge \Gamma \vdash \mathsf{v} : \sigma \wedge \cdot;\Gamma^{+} \vdash \mathbf{v} : \sigma^{+} \wedge \\
&\qquad \forall (k, \gamma, \boldsymbol{\gamma}) \in \mathcal{G}^{+}[\![\Gamma]\!].(k, \gamma(\mathsf{v}), \boldsymbol{\gamma}(\mathbf{v})) \in \mathcal{V}^{+}[\![\sigma]\!] \\
\Gamma \vdash \mathsf{e} \approx_{\div} \mathbf{e} : \sigma \quad &\stackrel{\text{def}}{=} \quad \mathsf{e} \in \lambda^{\mathrm{S}} \wedge \mathbf{e} \in \lambda^{\mathrm{T}} \wedge \Gamma \vdash \mathsf{e} : \sigma \wedge \cdot;\Gamma^{+} \vdash \mathbf{e} : \sigma^{\div} \wedge \\
&\qquad \forall (k, \gamma, \boldsymbol{\gamma}) \in \mathcal{G}^{+}[\![\Gamma]\!].(k, \gamma(\mathsf{e}), \boldsymbol{\gamma}(\mathbf{e})) \in \mathcal{E}^{\div}[\![\sigma]\!]
\end{aligned}
$$

Figure 20: Cross Language (λ^{ST}) Logical Relations for Closure Conversion Semantics Preservation

7 λ^{ST} Logical Relation Corresponds to Contextual Equivalence

7.1 λ^{ST} Logical Relation: Fundamental Property

Unless otherwise specified, all of the following lemmas additionally assume $\rho \in \mathcal{D}\llbracket\Delta\rrbracket$ and $\Delta \vdash \Gamma$.

Lemma 7.1 (Unique Decomposition)
If $\cdot;\cdot \vdash K[e] : \theta$ and $e \longmapsto e'$, then $K[e] \longmapsto K[e']$.

Proof

Omitted, standard. $\square$

Lemma 7.2 (Compositionality of Typing)
If $\rho \in \mathcal{D}\llbracket\Delta\rrbracket, (k,\gamma) \in \mathcal{G}\llbracket\Gamma\rrbracket\,\rho$ and $\Delta;\Gamma \vdash e : \sigma$, then $\cdot;\cdot \vdash \rho_1(\gamma_1(e)) : \rho_1(\theta)$ and $\cdot;\cdot \vdash \rho_2(\gamma_2(e)) : \rho_2(\theta)$

Proof

Omitted, standard. $\square$

Lemma 7.3 (Admissibility of Value Relation)
If $\rho \in \mathcal{D}\llbracket\Delta\rrbracket$ and $\Delta \vdash \boldsymbol{\tau}$, then $\mathcal{V}\llbracket\boldsymbol{\tau}\rrbracket\,\rho \in \mathrm{Rel}[\rho_1(\boldsymbol{\tau}), \rho_2(\boldsymbol{\tau})]$

Proof

Omitted. $\square$

Lemma 7.4 (Weakening of Logical Relations)
If $\rho \in \mathcal{D}\llbracket\Delta\rrbracket$ $(\Delta \vdash \tau), (\Delta \vdash \theta)$, $R \in \mathrm{Rel}[\boldsymbol{\tau_1}, \boldsymbol{\tau_2}]$ and $\rho' = \rho[\boldsymbol{\alpha} \mapsto (\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, R)]$ then

1. $\mathcal{V}\llbracket\tau\rrbracket\,\rho' = \mathcal{V}\llbracket\tau\rrbracket\,\rho$
2. $\mathcal{E}\llbracket\theta\rrbracket\,\rho' = \mathcal{E}\llbracket\theta\rrbracket\,\rho$
3. $\mathcal{R}\llbracket\theta\rrbracket\rho' = \mathcal{R}\llbracket\theta\rrbracket\rho$
4. $\mathcal{K}\llbracket\theta\rrbracket\,\rho' = \mathcal{K}\llbracket\theta\rrbracket\,\rho$

Proof

By mutual induction on $k, \Delta \vdash \tau, \Delta \vdash \theta$. $\square$

Lemma 7.5 (Compositionality of Logical Relations)
If $\rho \in \mathcal{D}\llbracket\Delta\rrbracket, (\Delta, \boldsymbol{\alpha} \vdash \tau), (\Delta \vdash \boldsymbol{\tau})$ and $\Delta, \boldsymbol{\alpha} \vdash \theta$ then

1. $(j, e_1, e_2) \in \mathcal{E}\llbracket\theta\rrbracket\,\rho'$ *if and only if* $(j, e_1, e_2) \in \mathcal{E}\llbracket\theta[\boldsymbol{\tau}/\boldsymbol{\alpha}]\rrbracket\,\rho$
2. $(j, e_1, e_2) \in \mathcal{V}\llbracket\tau\rrbracket\,\rho'$ *if and only if* $(j, e_1, e_2) \in \mathcal{V}\llbracket\tau[\boldsymbol{\tau}/\boldsymbol{\alpha}]\rrbracket\,\rho$

where $\rho' = \rho[\boldsymbol{\alpha} \mapsto (\rho_1(\boldsymbol{\tau}), \rho_2(\boldsymbol{\tau}), \mathcal{V}\llbracket\tau\rrbracket\,\rho)]$

Proof

By mutual induction on $k, \Delta, \boldsymbol{\alpha} \vdash \tau, \Delta, \boldsymbol{\alpha} \vdash \theta$. $\square$

Lemma 7.6 (Monotonicity of Value Relation)
If $j, k \in \mathbb{N}$, $j \leq k$, and $(k, v_1, v_2) \in \mathcal{V}\llbracket\tau\rrbracket\,\rho$ then $(j, v_1, v_2) \in \mathcal{V}\llbracket\tau\rrbracket\,\rho$.

Proof

By induction on τ.

Case 1, immediate.

Case $\sigma_1 + \sigma_2$ by inductive hypothesis.

Case $\sigma_1 \times \sigma_2$ by inductive hypothesis.

Case $\sigma \to \sigma'$, by transitivity of $\leq$.

Case $\mu\alpha.\,\sigma$, by transitivity of $<$.

Case $\mathbf{0}$, vacuously true.

Case $\langle\overline{\tau}\rangle$, by inductive hypothesis.

Case $\boldsymbol{\tau_1 + \tau_2}$ by inductive hypothesis.

Case $\boldsymbol{\alpha}$, by definition of Rel and $\rho_R(\boldsymbol{\alpha}) \in \mathrm{Rel}[\boldsymbol{\tau_1}, \boldsymbol{\tau_2}]$ for some $\boldsymbol{\tau_1}, \boldsymbol{\tau_2}$ since $\rho \in \mathcal{D}\,[\![\Delta]\!]$.

Case $\forall[\boldsymbol{\alpha}].\,\boldsymbol{\tau} \to \mathbf{E}\,\boldsymbol{\tau_{\mathrm{exn}}}\,\boldsymbol{\tau}'$, by transitivity of $\leq$.

Case $\boldsymbol{\mu\alpha.\,\tau}$, by transitivity of $<$.

Case $\boldsymbol{\exists\alpha.\,\tau}$, by inductive hypothesis.

$\square$

Lemma 7.7 (Monotonicity of G Relation)
If $j, k \in \mathbb{N}, j \leq k$, and $(k, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$ then $(j, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$.

Proof

By induction on structure of γ, and Lemma 7.6. $\square$

Lemma 7.8 (Result Relation Embeds in Expression Relation)
$\mathcal{R}[\![\theta]\!]\rho \subset \mathcal{E}\,[\![\theta]\!]\,\rho$.

Proof

Immediate by definition of $\mathcal{K}\,[\![\theta]\!]\,\rho$. $\square$

Lemma 7.9 (Monadic Bind)
If $(k, e_1, e_2) \in \mathcal{E}\,[\![\theta]\!]\,\rho$
and $(\forall j \leq k, r_1, r_2, (j, r_1, r_2) \in \mathcal{R}[\![\theta]\!]\rho \implies (j, K_1[r_1], K_2[r_2]) \in \mathcal{E}\,[\![\theta]\!]\,\rho)$,
then $(k, K_1[e_1], K_2[e_2]) \in \mathcal{E}\,[\![\theta]\!]\,\rho$.

Proof

Suppose $(k, K_1', K_2') \in \mathcal{K}\,[\![\theta]\!]\,\rho$, we need to show that $(k, K_1'[K_1[e_1]], K_2'[K_2[e_2]]) \in \mathcal{O}$.

Since $(k, e_1, e_2) \in \mathcal{E}\,[\![\theta]\!]\,\rho$, it is sufficient to show that $(k, K_1'[K_1], K_2'[K_2]) \in \mathcal{K}\,[\![\theta]\!]\,\rho$.

Suppose $j \leq k, (j, r_1, r_2) \in \mathcal{R}[\![\theta]\!]\rho$, we seek to prove that $(j, K_1'[K_1[r_1]], K_2'[K_2[r_2]]) \in \mathcal{O}$.

By hypothesis, $(j, K_1[r_1], K_2[r_2]) \in \mathcal{E}\,[\![\theta]\!]\,\rho$, so $(j, K_1'[K_1[r_1]], K_2'[K_2[r_2]]) \in \mathcal{O}$ by definition of $\mathcal{E}\,[\![\theta]\!]\,\rho$. $\square$

Lemma 7.10 (Observation Relation closed under Anti-Reduction)
If $e_1 \longmapsto^{k_1} e_1'$, $e_2 \longmapsto^{k_2} e_2'$ and $(k', e_1', e_2') \in \mathcal{O}$
then for any $0 \leq k \leq k' + \min(k_1, k_2)$, $(k, e_1, e_2) \in \mathcal{O}$.

Proof

If $e_1' \Downarrow \;\wedge\; e_2' \Downarrow$, then $e_1 \Downarrow \;\wedge\; e_2 \Downarrow$.

Otherwise we know that there exist e_1', e_2' such that $e_1' \longmapsto^{k'+1} e_1''$ and $e_2' \longmapsto^{k'+1} e_2''$.

Thus $e_1 \longmapsto^{k'+k_1+1} e_1''$ and $e_2 \longmapsto^{k'+k_2+1} e_2''$, and since $k \leq k' + k_1$, $k + 1 \leq k' + k_1 + 1$ and similarly $k + 1 \leq k' + k_2 + 1$ there must exist e_1''', e_2''' such that $e_1 \longmapsto^{k+1} e_1'''$ and $e_2 \longmapsto^{k+1} e_2'''$, so $(k, e_1, e_2) \in \mathcal{O}$. $\square$

Lemma 7.11 (Expression Relation closed under Anti-Reduction)
If $(k, e_1, e_2) \in \mathrm{Atom}[\theta]\rho$, $e_1 \longmapsto^{k_1} e_1'$, $e_2 \longmapsto^{k_2} e_2'$, $(k', e_1', e_2') \in \mathcal{E}\,[\![\theta]\!]\,\rho$ *and* $k \leq k' + \min(k_1, k_2)$ *then* $(k, e_1, e_2) \in \mathcal{E}\,[\![\theta]\!]\,\rho$.

Proof

By definition of $\mathcal{E}$, Lemma 7.1 and Lemma 7.10. □

Lemma 7.12 (Compatibility Source Var)
If $\mathsf{x} : \sigma \in \Gamma$ *and* $\Delta \vdash \Gamma$ *then* $\Delta; \Gamma \vdash \mathsf{x} \approx_{\mathcal{V}}^{log} \mathsf{x} : \sigma$.

Proof

$\Delta; \Gamma \vdash \mathsf{x} : \sigma$ by definition of the type system.

Suppose $\rho \in \mathcal{D}\,[\![\Delta]\!]\,, (k, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$. Then, by definition of $\mathcal{D}, \mathcal{G}$, $(k, \rho_1(\gamma_1(\mathsf{x})), \rho_2(\gamma_2(\mathsf{x}))) \in \mathcal{V}\,[\![\theta]\!]\,\rho$. □

Lemma 7.13 (Compatibility Source Unit)
$\Delta; \Gamma \vdash \langle\rangle \approx_{\mathcal{V}}^{log} \langle\rangle : 1$

Proof

Immediate by definition of $\mathcal{V}\,[\![1]\!]\,\rho$. □

Lemma 7.14 (Compatibility Source Sum)
If $\Delta; \Gamma \vdash \mathsf{v}_1 \approx_{\mathcal{V}}^{log} \mathsf{v}_2 : \sigma_i$ *then* $\Delta; \Gamma \vdash \mathsf{inj_i}\, \mathsf{v}_1 \approx_{\mathcal{V}}^{log} \mathsf{inj_i}\, \mathsf{v}_2 : \sigma_1 + \sigma_2$

Proof

Standard. □

Lemma 7.15 (Compatibility Source Case)
If $\Delta; \Gamma \vdash \mathsf{v}_1 \approx_{\mathcal{V}}^{log} \mathsf{v}_2 : \sigma_1 + \sigma_2$, $\Delta; \Gamma, \mathsf{x} : \sigma_1 \vdash \mathsf{e}_1 \approx_{\mathcal{E}}^{log} \mathsf{e}_1' : \sigma$, *and* $\Delta; \Gamma, \mathsf{y} : \sigma_2 \vdash \mathsf{e}_2 \approx_{\mathcal{E}}^{log} \mathsf{e}_2' : \sigma$ *then* $\Delta; \Gamma \vdash \mathsf{case}\, \mathsf{v}_1\, \mathsf{of}\, \mathsf{x}.\, \mathsf{e}_1 \mid \mathsf{y}.\, \mathsf{e}_2 \approx_{\mathcal{E}}^{log} \mathsf{case}\, \mathsf{v}_2\, \mathsf{of}\, \mathsf{x}.\, \mathsf{e}_1' \mid \mathsf{y}.\, \mathsf{e}_2' : \sigma$.

Proof

Standard. □

Lemma 7.16 (Compatibility Source Pair)
If $\Delta; \Gamma \vdash \mathsf{v}_1 \approx_{\mathcal{V}}^{log} \mathsf{v}_1' : \sigma_1$ *and* $\Delta; \Gamma \vdash \mathsf{v}_2 \approx_{\mathcal{V}}^{log} \mathsf{v}_2' : \sigma_2$ *then* $\Delta; \Gamma \vdash \langle \mathsf{v}_1, \mathsf{v}_2 \rangle \approx_{\mathcal{V}}^{log} \langle \mathsf{v}_2, \mathsf{v}_2' \rangle : \sigma_1 \times \sigma_2$

Proof

Standard. □

Lemma 7.17 (Compatibility Source Projection)
If $\Delta; \Gamma \vdash \mathsf{v} \approx_{\mathcal{V}}^{log} \mathsf{v}' : \sigma_1 \times \sigma_2$ *then* $\Delta; \Gamma \vdash \pi_{\mathsf{i}}\, \mathsf{v} \approx_{\mathcal{E}}^{log} \pi_{\mathsf{i}}\, \mathsf{v}' : \sigma_i$

Proof

Standard. □

Lemma 7.18 (Compatibility Source Abs)
If $\Delta; \Gamma, \mathsf{x} : \sigma \vdash \mathsf{e}_1 \approx_{\mathcal{E}}^{log} \mathsf{e}_2 : \sigma'$
then $\Delta; \Gamma \vdash \lambda(\mathsf{x} : \sigma).\, \mathsf{e}_1 \approx_{\mathcal{V}}^{log} \lambda(\mathsf{x} : \sigma).\, \mathsf{e}_2 : \sigma \to \sigma'$.

Proof

Suppose $k \geq 0, \rho \in \mathcal{D}[\![\Delta]\!], (k, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. Then $(k, \rho_1(\gamma_1(\lambda(x : \sigma).\, e_1)), \rho_2(\gamma_2(\lambda(x : \sigma).\, e_2))) \in \text{Atom}[\sigma \to \sigma', \sigma \to \sigma']$ by Lemma 7.2.

Suppose $j \leq k, (j, v_1, v_2) \in \mathcal{V}[\![\sigma]\!]\rho$. We need to show that $(j, \rho_1(\gamma_1(e_1))[v_1/x], \rho_2(\gamma_2(e_2))[v_2/x]) \in \mathcal{E}[\![\sigma']\!]\rho$.

Let $\gamma' = \gamma[x \mapsto (v_1, v_2)]$, then by hypothesis, it is sufficient to show that $(j, \gamma') \in \mathcal{G}[\![\Gamma, x : \sigma]\!]\rho$. This holds by assumption that $(j, v_1, v_2) \in \mathcal{V}[\![\sigma]\!]\rho$ and Lemma 7.7. □

Lemma 7.19 (Compatibility Source App)
If $\Delta; \Gamma \vdash v_1 \approx^{log}_{\mathcal{V}} v_2 : \sigma \to \sigma'$ *and* $\Delta; \Gamma \vdash v_1' \approx^{log}_{\mathcal{V}} v_2' : \sigma$
then $\Delta; \Gamma \vdash v_1\ v_1' \approx^{log}_{\mathcal{E}} v_2\ v_2' : \sigma'$.

Proof

Direct from definition of value relation at function type and Lemma 7.11. □

Lemma 7.20 (Compatibility Source Fold)
If $\Delta; \Gamma \vdash v_1 \approx^{log}_{\mathcal{V}} v_2 : \sigma[\mu\alpha.\, \sigma/\alpha]$
then $\Delta; \Gamma \vdash \mathsf{fold}_{\mu\alpha.\sigma}\, v_1 \approx^{log}_{\mathcal{V}} \mathsf{fold}_{\mu\alpha.\sigma}\, v_2 : \mu\alpha.\, \sigma$.

Proof

Direct from definition of value relation and Lemma 7.6. □

Lemma 7.21 (Compatibility Source Unfold)
If $\Delta; \Gamma \vdash v_1 \approx^{log}_{\mathcal{V}} v_2 : \mu\alpha.\, \sigma$
then $\Delta; \Gamma \vdash \mathsf{unfold}\, v_1 \approx^{log}_{\mathcal{E}} \mathsf{unfold}\, v_2 : \sigma[\mu\alpha.\, \sigma/\sigma]$.

Proof

Direct from definition of value relation and hypothesis. □

Lemma 7.22 (Compatibility Source Let)
If $\Delta; \Gamma \vdash e_1 \approx^{log}_{\mathcal{E}} e_2 : \sigma_1$ *and* $\Delta; \Gamma, x : \sigma_1 \vdash e_1' \approx^{log}_{\mathcal{E}} e_2' : \sigma_2$
then $\Delta; \Gamma \vdash \mathsf{let}\, x = e_1\, \mathsf{in}\, e_1' \approx^{log}_{\mathcal{E}} \mathsf{let}\, x = e_2\, \mathsf{in}\, e_2' : \sigma_2$.

Proof

Suppose $k \geq 0, \rho \in \mathcal{D}[\![\Delta]\!], (k, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$.

We seek to prove that $(k, \rho_1(\gamma_1(\mathsf{let}\, x = e_1\, \mathsf{in}\, e_1')), \rho_2(\gamma_2(\mathsf{let}\, x = e_2\, \mathsf{in}\, e_2'))) \in \mathcal{E}[\![\sigma_2]\!]\rho$.

By Lemma 7.9, it is sufficient to show that for any $j \leq k, v_1, v_2$, if $(j, v_1, v_2) \in \mathcal{V}[\![\sigma_1]\!]\rho$, then $(j, \mathsf{let}\, x = v_1\, \mathsf{in}\, \rho_1(\gamma_1(e_1'))\mathsf{let}\, x = v_2\, \mathsf{in}\, \rho_2(\gamma_2(e_2'))) \in \mathcal{E}[\![\sigma_2]\!]\rho$.

This holds by the fact that $(j, \gamma[x \mapsto (v_1, v_2)]) \in \mathcal{G}[\![\sigma_1]\!]\rho$ as in the proof of Lemma 7.18. □

Lemma 7.23 (Compatibility Target Var)
If $\mathbf{x} : \boldsymbol{\tau} \in \Gamma$ *and* $\Delta \vdash \Gamma$ *then* $\Delta; \Gamma \vdash \mathbf{x} \approx^{log}_{\mathcal{V}} \mathbf{x} : \boldsymbol{\tau}$.

Proof

Analogous to proof of Lemma 7.12 □

Lemma 7.24 (Compatibility Target Sum)
If $\Delta; \Gamma \vdash \mathbf{v} \approx^{log}_{\mathcal{V}} \mathbf{v}' : \boldsymbol{\tau}_i$ *then* $\Delta; \Gamma \vdash \mathbf{inj_i}\, \mathbf{v} \approx^{log}_{\mathcal{V}} \mathbf{inj_i}\, \mathbf{v}' : \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2$

Proof

Standard. □

Lemma 7.25 (Compatibility Target Case)
If $\Delta;\Gamma \vdash \mathbf{v} \approx^{log}_{\mathcal{V}} \mathbf{v'} : \boldsymbol{\tau_1 + \tau_2}$, $\Delta;\Gamma, \mathbf{x} : \boldsymbol{\tau_1} \vdash \mathbf{e_1} \approx^{log}_{\mathcal{E}} \mathbf{e'_1} : \boldsymbol{\theta}$, $\Delta;\Gamma, \mathbf{y} : \boldsymbol{\tau_2} \vdash \mathbf{e_2} \approx^{log}_{\mathcal{E}} \mathbf{e'_2} : \boldsymbol{\theta}$, *then* $\Delta;\Gamma \vdash \mathbf{case\,v\,of\,x.\,e_1 \mid y.\,e_2} \approx^{log}_{\mathcal{E}} \mathbf{case\,v'\,of\,x.\,e'_1 \mid y.\,e'_2} : \boldsymbol{\theta}$

Proof

Standard. □

Lemma 7.26 (Compatibility Target Tuple)
If $n \geq 0$, $\forall i \in \{1 \ldots n\}$. $\Delta;\Gamma \vdash \mathbf{v_{1,i}} \approx^{log}_{\mathcal{V}} \mathbf{v_{2,i}} : \boldsymbol{\tau_i}$
then $\Delta;\Gamma \vdash \langle \mathbf{v_{1,1}}, \ldots, \mathbf{v_{1,n}} \rangle \approx^{log}_{\mathcal{V}} \langle \mathbf{v_{2,1}}, \ldots, \mathbf{v_{2,n}} \rangle : \langle \boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_n} \rangle$.

Proof

Direct from definition of $\mathcal{V}\,[\![\langle \boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_n} \rangle]\!]\,\rho$. □

Lemma 7.27 (Compatibility Target Projection)
If $\Delta;\Gamma \vdash \mathbf{v_1} \approx^{log}_{\mathcal{V}} \mathbf{v_2} : \langle \boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_n} \rangle$,
then for any $i \in \{1, \ldots, n\}$, $\Delta;\Gamma \vdash \mathbf{return}_{\boldsymbol{\tau_{exn}}}\ \mathbf{v_1.i} \approx^{log}_{\mathcal{E}} \mathbf{return}_{\boldsymbol{\tau_{exn}}}\ \mathbf{v_2.i} : \mathbf{E}\,\boldsymbol{\tau_{exn}}\,\boldsymbol{\tau_i}$.

Proof

Suppose $k \geq 0, \rho \in \mathcal{D}\,[\![\Delta]\!]\,, (k, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$.

We seek to prove that $(k, \mathbf{return}\ (\rho_1(\gamma_1(\mathbf{v_1}))).\mathbf{i}, \mathbf{return}\ (\rho_2(\gamma_2(\mathbf{v_2}))).\mathbf{i}) \in \mathcal{E}\,[\![\mathbf{E}\,\boldsymbol{\tau_{exn}}\,\boldsymbol{\tau_i}]\!]\,\rho$.

By assumption, $(k, \rho_1(\gamma_1(\mathbf{v_1})), \rho_2(\gamma_2(\mathbf{v_2}))) \in \mathcal{V}\,[\![\langle \boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_n} \rangle]\!]\,\rho$, so $\rho_1(\gamma_1(\mathbf{v_1})) = \langle \mathbf{v_{1,1}}, \ldots, \mathbf{v_{1,n}} \rangle$ and $\rho_2(\gamma_2(\mathbf{v_2})) = \langle \mathbf{v_{2,1}}, \ldots, \mathbf{v_{2,n}} \rangle$, where importantly $(k, \mathbf{v_{1,i}} \mathbf{v_{2,i}}) \in \mathcal{V}\,[\![\boldsymbol{\tau_i}]\!]\,\rho$.

Next, $\mathbf{return}\ (\rho_1(\gamma_1(\mathbf{v_1}))) \longmapsto \mathbf{return}\ \mathbf{v_{1,i}}$ and $\mathbf{return}\ \rho_2(\gamma_2(\mathbf{v_2})) \longmapsto \mathbf{v_{2,i}}$. So by Lemma 7.11, it is sufficient to show $(k-1, \mathbf{return}\ \mathbf{v_{1,i}}, \mathbf{return}\ \mathbf{v_{2,i}}) \in \mathcal{E}\,[\![\mathbf{E}\,\boldsymbol{\tau_{exn}}\,\boldsymbol{\tau_i}]\!]\,\rho$, which follows from Lemma 7.8. □

Lemma 7.28 (Compatibility Target Abs)
If $\Delta, \boldsymbol{\alpha}; \Gamma, \mathbf{x} : \boldsymbol{\tau} \vdash \mathbf{e_1} \approx^{log}_{\mathcal{E}} \mathbf{e_2} : \boldsymbol{\theta}$
then $\Delta;\Gamma \vdash \boldsymbol{\lambda[\alpha](x : \tau).\, e_1} \approx^{log}_{\mathcal{V}} \boldsymbol{\lambda[\alpha](x : \tau).\, e_2} : \forall[\boldsymbol{\alpha}].\,\boldsymbol{\tau} \to \boldsymbol{\theta}$.

Proof

Suppose $k \geq 0, \rho \in \mathcal{D}\,[\![\Delta]\!]\,, (k, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$.

We need to show that $(k, \boldsymbol{\lambda[\alpha](x : \tau).}\, \rho_1(\gamma_1(\mathbf{e_1})), \boldsymbol{\lambda[\alpha](x : \tau).}\, \rho_2(\gamma_2(\mathbf{e_2}))) \in \mathcal{V}\,[\![\forall[\boldsymbol{\alpha}].\,\boldsymbol{\tau} \to \boldsymbol{\theta}]\!]\,\rho$.

Suppose $\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, R \in \mathrm{Rel}[\boldsymbol{\tau_1}, \boldsymbol{\tau_2}], j \leq k, (j, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}\,[\![\boldsymbol{\tau}]\!]\,\rho[\boldsymbol{\alpha} \mapsto (\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, R)]$. We need to show that $(j, \rho_1(\gamma_1(\mathbf{e_1}))[\boldsymbol{\tau_1}/\boldsymbol{\alpha}][\mathbf{v_1}/\mathbf{x}], \rho_2(\gamma_2(\mathbf{e_2}))[\boldsymbol{\tau_2}/\boldsymbol{\alpha}][\mathbf{v_2}/\mathbf{x}]) \in \mathcal{E}\,[\![\boldsymbol{\theta}]\!]\,\rho[\boldsymbol{\alpha} \mapsto (\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, R)]$

If we define $\rho' = \rho[\boldsymbol{\alpha} \mapsto (\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, R)]$ and $\gamma' = \gamma[\mathbf{x} \mapsto (\mathbf{v_1}, \mathbf{v_2})]$, then $\rho_1(\gamma_1(\mathbf{e_1}))[\boldsymbol{\tau_1}/\boldsymbol{\alpha}][\mathbf{v_1}/\mathbf{x}] = {\rho'}_1(\gamma'_1(\mathbf{e_1}))$ and $\rho_1(\gamma_1(\mathbf{e_2}))[\boldsymbol{\tau_2}/\boldsymbol{\alpha}][\mathbf{v_2}/\mathbf{x}] = {\rho'}_2(\gamma'_2(\mathbf{e_2}))$.

Furthermore, $\rho' \in \mathcal{D}\,[\![\Delta, \boldsymbol{\alpha}]\!]$ and $\gamma' \in \mathcal{G}\,[\![\Gamma, \mathbf{x} : \boldsymbol{\tau}]\!]$, which with our hypothesis gives us our goal $(j, {\rho'}_1(\gamma'_1(\mathbf{e_1})), {\rho'}_2(\gamma'_2(\mathbf{e_2}))) \in \mathcal{E}\,[\![\boldsymbol{\theta}]\!]\,\rho'$. □

Lemma 7.29 (Compatibility Target App)
If $\Delta \vdash \boldsymbol{\tau'}$ *and* $\Delta;\Gamma \vdash \mathbf{v_1} \approx^{log}_{\mathcal{V}} \mathbf{v_2} : \forall[\boldsymbol{\alpha}].\,\boldsymbol{\tau} \to \boldsymbol{\theta}$ *and* $\Delta;\Gamma \vdash \mathbf{v'_1} \approx^{log}_{\mathcal{V}} \mathbf{v'_2} : \boldsymbol{\tau}[\boldsymbol{\tau'}/\boldsymbol{\alpha}]$
then $\Delta;\Gamma \vdash \mathbf{v_1}\ [\boldsymbol{\tau'}]\ \mathbf{v'_1} \approx^{log}_{\mathcal{E}} \mathbf{v_2}\ [\boldsymbol{\tau'}]\ \mathbf{v'_2} : \boldsymbol{\theta}[\boldsymbol{\tau'}/\boldsymbol{\alpha}]$.

Proof

Suppose $k \geq 0, \rho \in \mathcal{D}\,[\![\Delta]\!]\,, (k, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$.
We need to show that $(k, \rho_1(\gamma_1(\mathbf{v_1}\ [\boldsymbol{\tau'}]\ \mathbf{v'_1})), \rho_2(\gamma_2(\mathbf{v_2}\ [\boldsymbol{\tau'}]\ \mathbf{v'_2}))) \in \mathcal{E}\,[\![\boldsymbol{\theta}[\boldsymbol{\tau'}/\boldsymbol{\alpha}]]\!]\,\rho$.

By definition of $\mathcal{V}\,[\![\forall[\boldsymbol{\alpha}].\,\boldsymbol{\tau} \to \boldsymbol{\theta}]\!]\,\rho$, $\rho_1(\gamma_1(\mathbf{v_1})) = \boldsymbol{\lambda[\alpha](x : \tau_1).\, e_1}$ and $\rho_2(\gamma_2(\mathbf{v_2})) = \boldsymbol{\lambda[\alpha](x : \tau_2).\, e_2}$.

Then $\rho_1(\gamma_1(\mathbf{v_1}\ [\boldsymbol{\tau}']\ \mathbf{v_1'})) \longmapsto \mathbf{e_1}[\rho_1(\boldsymbol{\tau}')/\boldsymbol{\alpha}][\rho_1(\gamma_1(\mathbf{v_1'}))/\mathbf{x}]$
and $\rho_2(\gamma_2(\mathbf{v_2}\ [\boldsymbol{\tau}']\ \mathbf{v_2'})) \longmapsto \mathbf{e_2}[\rho_2(\boldsymbol{\tau}')/\boldsymbol{\alpha}][\rho_2(\gamma_2(\mathbf{v_2'}))/\mathbf{x}]$

Then by Lemma 7.11, it is sufficient to show that for $j < k$,
$(k-1, \mathbf{e_1}[\rho_1(\boldsymbol{\tau}')/\boldsymbol{\alpha}][\rho_1(\gamma_1(\mathbf{v_1'}))/\mathbf{x}], \mathbf{e_2}[\rho_2(\boldsymbol{\tau}')/\boldsymbol{\alpha}][\rho_2(\gamma_2(\mathbf{v_2'}))/\mathbf{x}]) \in \mathcal{E}\,[\![\boldsymbol{\theta}[\boldsymbol{\tau}'/\boldsymbol{\alpha}]]\!]\,\rho$.

Define $\rho' = \rho[\boldsymbol{\alpha} \mapsto (\rho_1(\boldsymbol{\tau}'), \rho_2(\boldsymbol{\tau}'), \mathcal{V}\,[\![\boldsymbol{\tau}']\!]\,\rho)]$. By Lemma 7.3, $\mathcal{V}\,[\![\boldsymbol{\tau}']\!]\,\rho \in \mathrm{Rel}[\rho_1(\boldsymbol{\tau}')][\rho_2(\boldsymbol{\tau}')]$, so $\rho' \in \mathcal{D}\,[\![\Delta, \boldsymbol{\alpha}]\!]$.

Then by Lemma 7.5 it is sufficient to show
$(k-1, \mathbf{e_1}[\rho_1(\boldsymbol{\tau}')/\boldsymbol{\alpha}][\rho_1(\gamma_1(\mathbf{v_1'}))/\mathbf{x}], \mathbf{e_2}[\rho_2(\boldsymbol{\tau}')/\boldsymbol{\alpha}][\rho_2(\gamma_2(\mathbf{v_2'}))/\mathbf{x}]) \in \mathcal{E}\,[\![\boldsymbol{\theta}]\!]\,\rho'$,
and so by definition of $\mathcal{V}\,[\![\forall[\boldsymbol{\alpha}].\,\boldsymbol{\tau} \to \boldsymbol{\theta}]\!]\,\rho$, it is sufficient to show that $(k-1, \rho_1(\gamma_1(\mathbf{v_1'})), \rho_2(\gamma_2(\mathbf{v_2'}))) \in \mathcal{V}\,[\![\boldsymbol{\tau}]\!]\,\rho'$, which follows from Lemma 7.5. □

Lemma 7.30 (Compatibility Target Fold)
If $\Delta; \Gamma \vdash \mathbf{v_1} \approx^{log}_{\mathcal{V}} \mathbf{v_2} : \boldsymbol{\tau}[\boldsymbol{\mu\alpha.\,\tau}/\boldsymbol{\alpha}]$
then $\Delta; \Gamma \vdash \mathbf{fold}_{\boldsymbol{\mu\alpha.\tau}}\,\mathbf{v_1} \approx^{log}_{\mathcal{V}} \mathbf{fold}_{\boldsymbol{\mu\alpha.\tau}}\,\mathbf{v_2} : \boldsymbol{\mu\alpha.\,\tau}$.

Proof

Analogous to proof of Lemma 7.20 □

Lemma 7.31 (Compatibility Target Unfold)
If $\Delta; \Gamma \vdash \mathbf{v_1} \approx^{log}_{\mathcal{V}} \mathbf{v_2} : \boldsymbol{\mu\alpha.\,\tau}$
then $\Delta; \Gamma \vdash \mathbf{return}_{\boldsymbol{\tau}_{\mathbf{exn}}}\ \mathbf{unfold}\,\mathbf{v_1} \approx^{log}_{\mathcal{E}} \mathbf{return}_{\boldsymbol{\tau}_{\mathbf{exn}}}\ \mathbf{unfold}\,\mathbf{v_2} : \mathbf{E}\,\boldsymbol{\tau}_{\mathbf{exn}}\,\boldsymbol{\tau}[\boldsymbol{\mu\alpha.\,\tau}/\boldsymbol{\tau}]$.

Proof

Suppose $k \geq 0, \rho \in \mathcal{D}\,[\![\Delta]\!]\,, (k, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho,$.
We need to show that $(k, \mathbf{return}\ (\mathbf{unfold}\,\rho_1(\gamma_1(\mathbf{v_1}))), \mathbf{return}\ (\mathbf{unfold}\,\rho_2(\gamma_2(\mathbf{v_2})))) \in \mathcal{E}\,[\![\mathbf{E}\,\boldsymbol{\tau}_{\mathbf{exn}}\,\boldsymbol{\tau}[\boldsymbol{\mu\alpha.\,\tau}/\boldsymbol{\tau}]]\!]\,\rho$.

By hypothesis and definition of $\mathcal{V}\,[\![\boldsymbol{\mu\alpha.\,\tau}]\!]\,\rho$, $\rho_1(\gamma_1(\mathbf{v_1})) = \mathbf{fold}_{\boldsymbol{\mu\alpha.\tau}}\,\mathbf{v_1'}$ and $\rho_2(\gamma_2(\mathbf{v_2})) = \mathbf{fold}_{\boldsymbol{\mu\alpha.\tau}}\,\mathbf{v_2'}$ where for all $j < k, (j, \mathbf{v_1'}, \mathbf{v_2'}) \in \mathcal{V}\,[\![\boldsymbol{\tau}[\boldsymbol{\mu\alpha.\,\tau}/\boldsymbol{\tau}]]\!]\,\rho$.

Therefore, $\mathbf{return}\ (\mathbf{unfold}\,\rho_1(\gamma_1(\mathbf{v_1}))) \longmapsto \mathbf{return}\ \mathbf{v_1'}$ and
$\mathbf{return}\ (\mathbf{unfold}\,\rho_2(\gamma_2(\mathbf{v_2}))) \longmapsto \mathbf{return}\ \mathbf{v_2'}$. Finally, for any $(k-1, \mathbf{return}\ \mathbf{v_1'}, \mathbf{return}\ \mathbf{v_2'}) \in \mathcal{E}\,[\![\mathbf{E}\,\boldsymbol{\tau}_{\mathbf{exn}}\,\boldsymbol{\tau}[\boldsymbol{\mu\alpha.\,\tau}/\boldsymbol{\tau}]]\!]\,\rho$ by hypothesis and Lemma 7.8, so the result holds by Lemma 7.11. □

Lemma 7.32 (Compatibility Target Pack)
If $\Delta; \Gamma \vdash \mathbf{v_1} \approx^{log}_{\mathcal{V}} \mathbf{v_2} : \boldsymbol{\tau}[\boldsymbol{\tau}'/\boldsymbol{\alpha}]$
then $\Delta; \Gamma \vdash \mathbf{pack}\,(\boldsymbol{\tau}', \mathbf{v_1})\ \mathbf{as}\ \exists\boldsymbol{\alpha}.\,\boldsymbol{\tau} \approx^{log}_{\mathcal{V}} \mathbf{pack}\,(\boldsymbol{\tau}', \mathbf{v_2})\ \mathbf{as}\ \exists\boldsymbol{\alpha}.\,\boldsymbol{\tau} : \exists\boldsymbol{\alpha}.\,\boldsymbol{\tau}$.

Proof

Suppose $k \geq 0, \rho \in \mathcal{D}\,[\![\Delta]\!]\,, (k, \gamma) \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$. We need to show that
$(k, \mathbf{pack}\,(\rho_1(\boldsymbol{\tau}'), \rho_1(\gamma_1(\mathbf{v_1})))\ \mathbf{as}\ \exists\boldsymbol{\alpha}.\,\rho_1(\boldsymbol{\tau}), \mathbf{pack}\,(\rho_2(\boldsymbol{\tau}'), \rho_2(\gamma_2(\mathbf{v_2})))\ \mathbf{as}\ \exists\boldsymbol{\alpha}.\,\rho_2(\boldsymbol{\tau})) \in \mathcal{V}\,[\![\exists\boldsymbol{\alpha}.\,\boldsymbol{\tau}]\!]\,\rho$.

First, by Lemma 7.3, $\mathcal{V}\,[\![\boldsymbol{\tau}']\!]\,\rho \in \mathrm{Rel}[\rho_1(\boldsymbol{\tau}'), \rho_2(\boldsymbol{\tau}')]$. Therefore it is sufficient to show that for any $j < k, (j, \rho_1(\gamma_1(\mathbf{v_1})), \rho_2(\gamma_2(\mathbf{v_2}))) \in \mathcal{V}\,[\![\boldsymbol{\tau}]\!]\,\rho[\boldsymbol{\alpha} \mapsto (\rho_1(\boldsymbol{\tau}'), \rho_2(\boldsymbol{\tau}'), \mathcal{V}\,[\![\boldsymbol{\tau}']\!]\,\rho)]$.

By Lemma 7.5, this is equivalent to showing $(j, \rho_1(\gamma_1(\mathbf{v_1})), \rho_2(\gamma_2(\mathbf{v_2}))) \in \mathcal{V}\,[\![\boldsymbol{\tau}[\boldsymbol{\tau}'/\boldsymbol{\alpha}]]\!]\,\rho$, which holds by hypothesis and Lemma 7.6. □

Lemma 7.33 (Compatibility Target Unpack)
If $\Delta; \Gamma \vdash \mathbf{v_1} \approx^{log}_{\mathcal{V}} \mathbf{v_2} : \exists\boldsymbol{\alpha}.\,\boldsymbol{\tau}$ *and* $\Delta, \boldsymbol{\alpha}; \Gamma, \mathbf{x} : \boldsymbol{\tau} \vdash \mathbf{e_1} \approx^{log}_{\mathcal{E}} \mathbf{e_2} : \boldsymbol{\theta}$
then $\Delta; \Gamma \vdash \mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{x}) = \mathbf{v_1}\ \mathbf{in}\,\mathbf{e_1} \approx^{log}_{\mathcal{E}} \mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{x}) = \mathbf{v_2}\ \mathbf{in}\,\mathbf{e_2} : \boldsymbol{\theta}$.

Proof

Suppose $k \geq 0, \rho \in \mathcal{D}[\![\Delta]\!], (k, \gamma) \in \mathcal{G}[\![\Gamma]\!]\, \rho$.
We need to show that $(k, \mathbf{unpack}\,(\alpha, \mathbf{x}) = \rho_1(\mathbf{v_1})\,\mathbf{in}\,\rho_1(\mathbf{e_1}), \mathbf{unpack}\,(\alpha, \mathbf{x}) = \rho_2(\mathbf{v_2})\,\mathbf{in}\,\rho_2(\mathbf{e_2})) \in \mathcal{E}[\![\theta]\!]\, \rho$.

By hypothesis and definition of $\mathcal{V}[\![\exists\alpha.\,\tau]\!]\, \rho$, $\rho_1(\mathbf{v_1}) = \mathbf{pack}\,(\mathbf{v}'_1, \tau_1)\,\mathbf{as}\,\exists\alpha.\,\rho_1(\tau)$ and $\rho_2(\mathbf{v_2}) = \mathbf{pack}\,(\mathbf{v}'_2, \tau_2)\,\mathbf{as}\,\exists\alpha.\,\rho_2(\tau)$, so $\mathbf{unpack}\,(\alpha, \mathbf{x}) = \rho_1(\mathbf{v_1})\,\mathbf{in}\,\rho_1(\mathbf{e_1}) \longmapsto \rho_1(\mathbf{e_1})[\tau_1/\alpha][\mathbf{v_1}/\mathbf{x}]$ and $\mathbf{unpack}\,(\alpha, \mathbf{x}) = \rho_2(\mathbf{v_2})\,\mathbf{i}$
$\rho_2(\mathbf{e_2})[\tau_2/\alpha][\mathbf{v_2}/\mathbf{x}]$.

Then the result holds by an analogous argument to that in the proof of Lemma 7.28. □

Lemma 7.34 (Compatibility Target Handle)
If $\Delta; \Gamma \vdash \mathbf{e_1} \approx^{log}_{\mathcal{E}} \mathbf{e_2} : \mathbf{E}\,\tau'_{\mathbf{exn}}\,\tau'$
and $\Delta; \Gamma, \mathbf{x} : \tau' \vdash \mathbf{e}'_1 \approx^{log}_{\mathcal{E}} \mathbf{e}'_2 : \mathbf{E}\,\tau_{\mathbf{exn}}\,\tau$
and $\Delta; \Gamma, \mathbf{y} : \tau'_{\mathbf{exn}} \vdash \mathbf{e}''_1 \approx^{log}_{\mathcal{E}} \mathbf{e}''_2 : \mathbf{E}\,\tau_{\mathbf{exn}}\,\tau$
then $\Delta; \Gamma \vdash \mathbf{handle}\,\mathbf{e_1}\,\mathbf{with}\,(\mathbf{x}.\,\mathbf{e}'_1)\,(\mathbf{y}.\,\mathbf{e}''_1) \approx^{log}_{\mathcal{E}} \mathbf{handle}\,\mathbf{e_2}\,\mathbf{with}\,(\mathbf{x}.\,\mathbf{e}'_2)\,(\mathbf{y}.\,\mathbf{e}''_2) : \mathbf{E}\,\tau_{\mathbf{exn}}\,\tau$.

Proof

Suppose $k \geq 0, \rho \in \mathcal{D}[\![\Delta]\!], (k, \gamma) \in \mathcal{G}[\![\Gamma]\!]\, \rho$.
We need to show that
$(k, \mathbf{handle}\,(\rho_1(\gamma_1(\mathbf{e_1})))\,\mathbf{with}\,(\mathbf{x}.\,\rho_1(\gamma_1(\mathbf{e}'_1)))\,(\mathbf{y}.\,\rho_1(\gamma_1(\mathbf{e}''_1))),$
$\mathbf{handle}\,(\rho_2(\gamma_2(\mathbf{e_2})))\,\mathbf{with}\,(\mathbf{x}.\,\rho_2(\gamma_2(\mathbf{e}'_2)))\,(\mathbf{y}.\,\rho_2(\gamma_2(\mathbf{e}''_2)))) \in \mathcal{E}[\![\mathbf{E}\,\tau_{\mathbf{exn}}\,\tau]\!]\, \rho$
Applying Lemma 7.9, there are two cases

1. Suppose $j \leq k, (j, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\tau']\!]\, \rho$, then we need to show that
$(j, \mathbf{handle}\,\mathbf{return}\,\mathbf{v_1}\,\mathbf{with}\,(\mathbf{x}.\,\rho_1(\gamma_1(\mathbf{e}'_1)))\,(\mathbf{y}.\,\rho_1(\gamma_1(\mathbf{e}''_1))),$
$\mathbf{handle}\,\mathbf{return}\,\mathbf{v_2}\,\mathbf{with}\,(\mathbf{x}.\,\rho_2(\gamma_2(\mathbf{e}'_2)))\,(\mathbf{y}.\,\rho_2(\gamma_2(\mathbf{e}''_2)))) \in \mathcal{E}[\![\mathbf{E}\,\tau_{\mathbf{exn}}\,\tau]\!]\, \rho$.
Then $\mathbf{handle}\,\mathbf{return}\,\mathbf{v_1}\,\mathbf{with}\,(\mathbf{x}.\,\rho_1(\gamma_1(\mathbf{e}'_1)))\,(\mathbf{y}.\,\rho_1(\gamma_1(\mathbf{e}''_1))) \longmapsto \rho_1(\gamma_1(\mathbf{e}'_1))[\mathbf{v_1}/\mathbf{x}]$ and
$\mathbf{handle}\,\mathbf{return}\,\mathbf{v_2}\,\mathbf{with}\,(\mathbf{x}.\,\rho_2(\gamma_2(\mathbf{e}'_2)))\,(\mathbf{y}.\,\rho_2(\gamma_2(\mathbf{e}''_2))) \longmapsto \rho_2(\gamma_2(\mathbf{e}'_2))[\mathbf{v_2}/\mathbf{x}]$.
Let $\gamma' = \gamma[\mathbf{x} \mapsto (\mathbf{v_1}, \mathbf{v_2})]$. Then $\rho_1(\gamma_1(\mathbf{e}'_1))[\mathbf{v_1}/\mathbf{x}] = \rho_1(\gamma'_1(\mathbf{e}'_1))$ and $\rho_2(\gamma_2(\mathbf{e}'_2))[\mathbf{v_2}/\mathbf{x}] = \rho_2(\gamma'_2(\mathbf{e}'_2))$.
Furthermore, $\gamma' \in \mathcal{G}[\![\Gamma, \mathbf{x} : \tau']\!]$, so by hypothesis $(j, \rho_1(\gamma'_1(\mathbf{e}'_1)), \rho_2(\gamma'_2(\mathbf{e}'_2))) \in \mathcal{E}[\![\mathbf{E}\,\tau_{\mathbf{exn}}\,\tau]\!]\, \rho$ The result then holds by Lemma 7.11.
2. Suppose $j \leq k, (j, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\tau'_{\mathbf{exn}}]\!]\, \rho$, then we need to show that
$(j, \mathbf{handle}\,\mathbf{raise}\,\mathbf{v_1}\,\mathbf{with}\,(\mathbf{x}.\,\rho_1(\gamma_1(\mathbf{e}'_1)))\,(\mathbf{y}.\,\rho_1(\gamma_1(\mathbf{e}''_1))),$
$\mathbf{handle}\,\mathbf{raise}\,\mathbf{v_2}\,\mathbf{with}\,(\mathbf{x}.\,\rho_2(\gamma_2(\mathbf{e}'_2)))\,(\mathbf{y}.\,\rho_2(\gamma_2(\mathbf{e}''_2)))) \in \mathcal{E}[\![\mathbf{E}\,\tau_{\mathbf{exn}}\,\tau]\!]\, \rho$.
Analogous to the previous case.

□

Lemma 7.35 (Bridge Lemmas)
Let $\rho \in \mathcal{D}[\![\Delta]\!], \Delta \vdash \sigma$.

1. *If* $(k, \mathbf{e_1}, \mathbf{e_2}) \in \mathcal{E}[\![\sigma^{\div}]\!]\, \rho$, *then* $(k, {}^{\sigma}\mathcal{ST}\,\mathbf{e_1}, {}^{\sigma}\mathcal{ST}\,\mathbf{e_2}) \in \mathcal{E}[\![\sigma]\!]\, \rho$.
2. *If* $(k, \mathbf{r_1}, \mathbf{r_2}) \in \mathcal{R}[\![\sigma^{\div}]\!]\rho$ *and* ${}^{\sigma}\mathcal{ST}\,\mathbf{r_1} \longmapsto^n \mathsf{v}_1$ *and* ${}^{\sigma}\mathcal{ST}\,\mathbf{r_2} \longmapsto^m \mathsf{v}_1$, *then* $(k, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{R}[\![\sigma]\!]\rho$.
3. *If* $(k, \mathsf{e}_1, \mathsf{e}_2) \in \mathcal{E}[\![\sigma]\!]\, \rho$, *then* $(k, \mathcal{TS}^{\,\sigma}\,\mathsf{e}_1, \mathcal{TS}^{\,\sigma}\,\mathsf{e}_2) \in \mathcal{E}[\![\sigma^{\div}]\!]\, \rho$.
4. *If* $(k, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{R}[\![\sigma]\!]\rho$ *and* $\mathcal{TS}^{\,\sigma}\,\mathsf{v}_1 \longmapsto^n \mathbf{r_1}$ *and* ${}^{\sigma}\mathcal{ST}\,\mathsf{v}_1 \longmapsto^m \mathbf{r_2}$, *then* $(k, \mathbf{r_1}, \mathbf{r_2}) \in \mathcal{R}[\![\sigma^{\div}]\!]\rho$.

Proof

Proved simultaneously by induction on σ, k.

1. By Lemma 7.9, it is sufficient to prove that for all $j \leq k, (j, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\sigma^{+}]\!]$,
$(j, {}^{\sigma}\mathcal{ST}\,\mathbf{return}\,\mathbf{v_1}, {}^{\sigma}\mathcal{ST}\,\mathbf{return}\,\mathbf{v_2}) \in \mathcal{E}[\![\sigma]\!]\, \rho$ and for all $j \leq k, (j, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\mathbf{0}]\!]\, \rho$,
$(j, {}^{\sigma}\mathcal{ST}\,\mathbf{raise}\,\mathbf{v_1}, {}^{\sigma}\mathcal{ST}\,\mathbf{raise}\,\mathbf{v_2}) \in \mathcal{E}[\![\sigma]\!]\, \rho$. The latter is vacuously true since $\mathcal{V}[\![\mathbf{0}]\!]\, \rho = \emptyset$.
For the former case, note that $(j, \mathbf{return}\,\mathbf{v_1}, \mathbf{return}\,\mathbf{v_2}) \in \mathcal{R}[\![\sigma^{\div}]\!]\rho$ by definition of $\mathcal{R}[\![\sigma^{\div}]\!]\rho$ and the assumption that $(j, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\sigma^{+}]\!]$. The goal follows by case 2 of this lemma, and Lemma 7.8.

2. By case analysis of σ. We omit the uninteresting cases such as $\sigma_1 + \sigma_2$ and $\sigma_1 \times \sigma_2$

   Case $\sigma = \sigma'' \to \sigma'$: then $\sigma^+ = \exists\boldsymbol{\alpha}.\ \langle(\langle\boldsymbol{\alpha}, {\sigma_1}^+\rangle \to {\sigma_2}^\div), \boldsymbol{\alpha}\rangle$.

   By definition of $\mathcal{V}$, this means $\mathbf{v_1} = \mathbf{pack}\,(\boldsymbol{\tau_1}, \langle\mathbf{v'_1}, \mathbf{v''_1}\rangle)\,\mathbf{as}\,(\sigma'' \to \sigma')^+$ and $\mathbf{v_2} = \mathbf{pack}\,(\boldsymbol{\tau_2}, \langle\mathbf{v'_2}, \mathbf{v''_2}\rangle)\,\mathbf{as}\,(\sigma'' \to \sigma')^+$, where there is some relation $R \in \mathrm{Rel}[\boldsymbol{\tau_1}, \boldsymbol{\tau_2}]$ such that $(k, \mathbf{v'_1}, \mathbf{v'_2}) \in \mathcal{V}\,[\![\langle\boldsymbol{\alpha}, {\sigma_1}^+\rangle \to {\sigma_2}^\div]\!]\,\rho'$ and $(k, \mathbf{v''_1}, \mathbf{v''_2}) \in \mathcal{V}\,[\![\boldsymbol{\alpha}]\!]\,\rho'$, where $\rho' = \rho[\boldsymbol{\alpha} \mapsto (\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, R)]$.

   Next, ${}^{\sigma''\to\sigma'}\mathcal{ST}\,\mathbf{return\ v_1} \longmapsto$

   $$\lambda(\mathsf{x} : \sigma'').\ {}^{\sigma'}\mathcal{ST}\left(\begin{array}{l}\mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{z}) = \mathbf{v_1}\ \mathbf{in}\ \mathbf{let}\ \mathbf{x_f} = \mathbf{return\ z.1\ in} \\ \qquad \mathbf{let}\ \mathbf{x_{env}} = \mathbf{return\ z.2\ in} \\ \qquad \mathbf{let}\ \mathbf{x} = \mathcal{TS}^{\,\sigma_1}\,\mathsf{x}\ \mathbf{in}\ \mathbf{x_f}\ [\boldsymbol{\alpha}]\ \langle\mathbf{x_{env}}, \mathbf{x}\rangle\end{array}\right)$$

   and ${}^{\sigma''\to\sigma'}\mathcal{ST}\,\mathbf{return\ v_2} \longmapsto$

   $$\lambda(\mathsf{x} : \sigma'').\ {}^{\sigma'}\mathcal{ST}\left(\begin{array}{l}\mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{z}) = \mathbf{v_2}\ \mathbf{in}\ \mathbf{let}\ \mathbf{x_f} = \mathbf{return\ z.1\ in} \\ \qquad \mathbf{let}\ \mathbf{x_{env}} = \mathbf{return\ z.2\ in} \\ \qquad \mathbf{let}\ \mathbf{x} = \mathcal{TS}^{\,\sigma_2}\,\mathsf{x}\ \mathbf{in}\ \mathbf{x_f}\ [\boldsymbol{\alpha}]\ \langle\mathbf{x_{env}}, \mathbf{x}\rangle\end{array}\right)$$

   Suppose $j \le k$ and $(j, \mathsf{v}'''_1, \mathsf{v}'''_2) \in \mathcal{V}\,[\![\sigma'']\!]\,\rho$. We need to show that

   $$\left(j, {}^{\sigma'}\mathcal{ST}\left(\begin{array}{l}\mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{z}) = \mathbf{v_1}\ \mathbf{in}\ \mathbf{let}\ \mathbf{x_f} = \mathbf{return\ z.1\ in} \\ \qquad \mathbf{let}\ \mathbf{x_{env}} = \mathbf{return\ z.2\ in} \\ \qquad \mathbf{let}\ \mathbf{x} = \mathcal{TS}^{\,\sigma_1}\,\mathsf{v}'''_1\ \mathbf{in}\ \mathbf{x_f}\ [\boldsymbol{\alpha}]\ \langle\mathbf{x_{env}}, \mathbf{x}\rangle\end{array}\right), {}^{\sigma'}\mathcal{ST}\left(\begin{array}{l}\mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{z}) = \mathbf{v_2}\ \mathbf{in}\ \mathbf{let}\ \mathbf{x_f} = \mathbf{return\ z.1\ in} \\ \qquad \mathbf{let}\ \mathbf{x_{env}} = \mathbf{return\ z.2\ in} \\ \qquad \mathbf{let}\ \mathbf{x} = \mathcal{TS}^{\,\sigma_2}\,\mathsf{v}'''_2\ \mathbf{in}\ \mathbf{x_f}\ [\boldsymbol{\alpha}]\ \langle\mathbf{x_{env}}, \mathbf{x}\rangle\end{array}\right)\right)$$

   $\in \mathcal{E}\,[\![\sigma']\!]\,\rho$.

   By inductive hypothesis and Lemma 10.5, there exist $(j, \mathbf{v}'''_1, \mathbf{v}'''_2) \in \mathcal{V}\left[\!\!\left[{\sigma''}^+\right]\!\!\right]\rho'$ such that $\mathcal{TS}^{\,\sigma''}\,\mathsf{v}'''_1 \longmapsto^{n'} \mathbf{return\ v'''_1}$ and similarly $\mathcal{TS}^{\,\sigma''}\,\mathsf{v}'''_2 \longmapsto^{m'} \mathbf{return\ v'''_2}$ for some n', m'.

   Then ${}^{\sigma'}\mathcal{ST}\left(\begin{array}{l}\mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{z}) = \mathbf{v_1}\ \mathbf{in}\ \mathbf{let}\ \mathbf{x_f} = \mathbf{return\ z.1\ in} \\ \qquad \mathbf{let}\ \mathbf{x_{env}} = \mathbf{return\ z.2\ in} \\ \qquad \mathbf{let}\ \mathbf{x} = \mathcal{TS}^{\,\sigma_1}\,\mathsf{v}'''_1\ \mathbf{in}\ \mathbf{x_f}\ [\boldsymbol{\alpha}]\ \langle\mathbf{x_{env}}, \mathbf{x}\rangle\end{array}\right) \longmapsto^{n'+5}$

   ${}^{\sigma'}\mathcal{ST}\,\mathbf{v'_1}\ [\boldsymbol{\tau_1}]\ \langle\mathbf{v''_1}, \mathbf{v'''_1}\rangle$

   and similarly ${}^{\sigma'}\mathcal{ST}\left(\begin{array}{l}\mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{z}) = \mathbf{v_2}\ \mathbf{in}\ \mathbf{let}\ \mathbf{x_f} = \mathbf{return\ z.1\ in} \\ \qquad \mathbf{let}\ \mathbf{x_{env}} = \mathbf{return\ z.2\ in} \\ \qquad \mathbf{let}\ \mathbf{x} = \mathcal{TS}^{\,\sigma_2}\,\mathsf{v}'''_2\ \mathbf{in}\ \mathbf{x_f}\ [\boldsymbol{\alpha}]\ \langle\mathbf{x_{env}}, \mathbf{x}\rangle\end{array}\right) \longmapsto^{m'+5}$

   ${}^{\sigma'}\mathcal{ST}\,\mathbf{v'_2}\ [\boldsymbol{\tau_2}]\ \langle\mathbf{v''_2}, \mathbf{v'''_2}\rangle$.

   The result then holds by inductive hypothesis, Lemma 7.11, and similar reasoning to Lemma 7.29 and Lemma 7.26.

   Case $\sigma = \mu\alpha.\,\sigma'$: then $\sigma^+ = \boldsymbol{\mu\alpha}.\,{\sigma'}^+$.

   By definition of $\mathcal{V}\left[\!\!\left[\boldsymbol{\mu\alpha}.\,{\sigma'}^+\right]\!\!\right]\rho$, $\mathbf{v_1} = \mathbf{fold}_{\boldsymbol{\mu\alpha}.{\sigma'}^+}\,\mathbf{v'_1}$ and $\mathbf{v_2} = \mathbf{fold}_{\boldsymbol{\mu\alpha}.{\sigma'}^+}\,\mathbf{v'_2}$ such that for every $j < k$, $(j, \mathbf{v'_1}, \mathbf{v'_2}) \in \mathcal{V}\left[\!\!\left[{\sigma'}^+[\boldsymbol{\mu\alpha}.\,{\sigma'}^+/\boldsymbol{\alpha}]\right]\!\!\right]$.

   Next, ${}^{\mu\alpha.\sigma'}\mathcal{ST}\,\mathbf{return\ v_1} \longmapsto \mathsf{let}\,\mathsf{x} = {}^{\sigma'[\mu\alpha.\sigma'/\alpha]}\mathcal{ST}\,\mathbf{return_0\ unfold\,v_1}\ \mathsf{in\ fold}_{\mu\alpha.\sigma'}\,\mathsf{x}$ and ${}^{\mu\alpha.\sigma'}\mathcal{ST}\,\mathbf{return\ v_2} \longmapsto \mathsf{let}\,\mathsf{x} = {}^{\sigma'[\mu\alpha.\sigma'/\alpha]}\mathcal{ST}\,\mathbf{return_0\ unfold\,v_2}\ \mathsf{in\ fold}_{\mu\alpha.\sigma'}\,\mathsf{x}$.

   Furthermore, by Lemma 10.5 and inductive hypothesis, ${}^{\sigma'[\mu\alpha.\sigma'/\alpha]}\mathcal{ST}\,\mathbf{return_0\ v'_1} \longmapsto^{n} \mathsf{v}'_1$ and similarly ${}^{\sigma'[\mu\alpha.\sigma'/\alpha]}\mathcal{ST}\,\mathbf{return_0\ v'_2} \longmapsto^{m} \mathsf{v}'_2$ and $(j, \mathsf{v}'_1, \mathsf{v}'_2) \in \mathcal{V}\,[\![\sigma'[\mu\alpha.\,\sigma'/\alpha]]\!]\,\rho$ for every $j < k$.

   Therefore $\mathsf{let}\,\mathsf{x} = {}^{\sigma'[\mu\alpha.\sigma'/\alpha]}\mathcal{ST}\,\mathbf{return_0\ unfold\,v_1}\ \mathsf{in\ fold}_{\mu\alpha.\sigma'}\,\mathsf{x} \longmapsto^{n+2} \mathsf{fold}_{\mu\alpha.\sigma'}\,\mathsf{v}'_1$ and $\mathsf{let}\,\mathsf{x} = {}^{\sigma'[\mu\alpha.\sigma'/\alpha]}\mathcal{ST}\,\mathbf{return_0\ unfold\,v_2}\ \mathsf{in\ fold}_{\mu\alpha.\sigma'}\,\mathsf{x} \longmapsto^{n+2} \mathsf{fold}_{\mu\alpha.\sigma'}\,\mathsf{v}'_2$ So we need to show that $(k, \mathsf{fold}_{\mu\alpha.\sigma'}\,\mathsf{v}'_1, \mathsf{fold}_{\mu\alpha.\sigma'}\,\mathsf{v}'_2) \in \mathcal{V}\,[\![\mu\alpha.\,\sigma']\!]\,\rho$, which holds by definition of $\mathcal{V}\,[\![\mu\alpha.\,\sigma']\!]\,\rho$ and what we know about $\mathsf{v}'_1, \mathsf{v}'_2$.

3. By Lemma 7.9, it is sufficient to prove that for all $j \le k$ if $(j, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}\,[\![\sigma]\!]\,\rho$ then $(j, \mathcal{TS}^{\,\sigma}\,\mathsf{v}_1, \mathcal{TS}^{\,\sigma}\,\mathsf{v}_2) \in \mathcal{E}\,[\![\sigma^\div]\!]\,\rho$. The result then holds by the value case and Lemma 7.8.

4. By case analysis of σ. We omit the uninteresting cases such as $\sigma_1 + \sigma_2$ and $\sigma_1 \times \sigma_2$

Case $\sigma = \sigma''\to\sigma'$: $\mathcal{TS}^{\sigma}\, v_1 \longmapsto \mathbf{return_0\ pack\ (1,\langle\lambda(z:\langle 1, \sigma''^{+}\rangle).\ \mathcal{TS}^{\sigma'}\left(\begin{array}{l}\mathsf{let}\ x = {}^{\sigma''}\mathcal{ST}\ \mathbf{return_0\ z.2}\ \mathsf{in}\\ \quad v_1'\ x\end{array}\right), \langle\rangle\rangle)\ as\ (\sigma_1 \to \sigma_2)^{+}}$

and similarly $\mathcal{TS}^{\sigma}\, v_2 \longmapsto \mathbf{return_0\ pack\ (1,\langle\lambda(z:\langle 1, \sigma''^{+}\rangle).\ \mathcal{TS}^{\sigma'}\left(\begin{array}{l}\mathsf{let}\ x = {}^{\sigma''}\mathcal{ST}\ \mathbf{return_0\ z.2}\ \mathsf{in}\\ \quad v_2'\ x\end{array}\right), \langle\rangle\rangle)\ as\ (\sigma_2 \to \sigma_2)^{+}}$

so we need to show that these **packs** are in $\mathcal{V}[\![\exists\alpha.\ \langle(\langle\alpha, \sigma_1{}^{+}\rangle \to \sigma_2{}^{\div}), \alpha\rangle]\!]\,\rho$.
For our relation we choose $\mathcal{V}[\![\mathbf{1}]\!]\,\rho$ (justified by Lemma 7.3). By definition of $\mathcal{V}$, it is sufficient to prove that for any $j \le k$ and $(j, \langle\langle\rangle, \mathbf{v_1''}\rangle, \langle\langle\rangle, \mathbf{v_2''}\rangle) \in \mathcal{V}[\![\langle\alpha, \sigma_1{}^{+}\rangle]\!]\,\rho'$,

$$\left(j, \mathcal{TS}^{\sigma'}\left(\begin{array}{l}\mathsf{let}\ x = {}^{\sigma''}\mathcal{ST}\ \mathbf{return_0}\ \langle\langle\rangle, \mathbf{v_1''}\rangle\mathbf{.2}\ \mathsf{in}\\ \quad v_1'\ x\end{array}\right), \mathcal{TS}^{\sigma'}\left(\begin{array}{l}\mathsf{let}\ x = {}^{\sigma''}\mathcal{ST}\ \mathbf{return_0}\ \langle\langle\rangle, \mathbf{v_2''}\rangle\mathbf{.2}\ \mathsf{in}\\ \quad v_2'\ x\end{array}\right)\right)$$

$\in \mathcal{E}[\![\sigma_2{}^{\div}]\!]\,\rho'$ where $\rho' = \rho[\alpha \mapsto (\mathbf{1}, \mathbf{1}, \mathcal{V}[\![\mathbf{1}]\!]\,\rho)]$.
By inductive hypothesis and Lemma 10.5, there exist v_1'', v_2'' such that $(j, v_1'', v_2'') \in \mathcal{V}[\![\sigma_1{}^{+}]\!]\,\rho'$, ${}^{\sigma''}\mathcal{ST}\ \mathbf{return\ v_1''} \longmapsto^{n'} v_1''$ and ${}^{\sigma''}\mathcal{ST}\ \mathbf{return\ v_2''} \longmapsto^{m'} v_2''$.

Then $\mathcal{TS}^{\sigma'}\left(\begin{array}{l}\mathsf{let}\ x = {}^{\sigma''}\mathcal{ST}\ \mathbf{return_0}\ \langle\langle\rangle, \mathbf{v_1''}\rangle\mathbf{.2}\ \mathsf{in}\\ \quad v_1'\ x\end{array}\right) \longmapsto^{n'+2} v_1'\ v_1''$ and

$\mathcal{TS}^{\sigma'}\left(\begin{array}{l}\mathsf{let}\ x = {}^{\sigma''}\mathcal{ST}\ \mathbf{return_0}\ \langle\langle\rangle, \mathbf{v_2''}\rangle\mathbf{.2}\ \mathsf{in}\\ \quad v_2'\ x\end{array}\right) \longmapsto^{m'+2} v_2'\ v_2''$. So the result holds by similar reasoning to Lemma 7.19.

Case $\sigma = \mu\alpha.\,\sigma'$: the proof follows similarly to the corresponding case above.

□

Lemma 7.36 **(Compatibility Source Boundary)**
If $\Delta; \Gamma \vdash \mathbf{e_1} \approx^{log}_{\mathcal{E}} \mathbf{e_2} : \sigma^{\div}$*, then* $\Delta; \Gamma \vdash {}^{\sigma}\mathcal{ST}\,\mathbf{e_1} \approx^{log}_{\mathcal{E}} {}^{\sigma}\mathcal{ST}\,\mathbf{e_2} : \sigma$*.*

Proof

Immediate by Lemma 7.35 □

Lemma 7.37 **(Compatibility Target Boundary)**
If $\Delta; \Gamma \vdash e_1 \approx^{log}_{\mathcal{E}} e_2 : \sigma$*, then* $\Delta; \Gamma \vdash \mathcal{TS}^{\sigma}\, e_1 \approx^{log}_{\mathcal{E}} \mathcal{TS}^{\sigma}\, e_2 : \sigma^{\div}$

Proof

Immediate by Lemma 7.35 □

Theorem 7.38 **(Fundamental Properties)**
The following are proved by mutual induction.

1. *If* $\Delta; \Gamma \vdash e : \theta$*, then* $\Delta; \Gamma \vdash e \approx^{log}_{\mathcal{E}} e : \theta$
2. *If* $\Delta; \Gamma \vdash v : \tau$*, then* $\Delta; \Gamma \vdash v \approx^{log}_{\mathcal{V}} v : \tau$

Proof

By induction on the typing derivation, then immediate by appropriate compatibility lemma. □

Lemma 7.39 **(Context Fundamental Property)**
There are four cases, depending on whether the context takes values or produces values.

1. *If* $\vdash C : (\Delta; \Gamma \vdash \theta) \Rightarrow (\Delta'; \Gamma' \vdash \theta')$, *then* $\vdash C \approx^{log}_{\mathcal{E}\Rightarrow\mathcal{E}} C : (\Delta; \Gamma \vdash \theta) \Rightarrow (\Delta'; \Gamma' \vdash \theta')$.
2. *If* $\vdash C : (\Delta; \Gamma \vdash \theta) \Rightarrow (\Delta'; \Gamma' \vdash \tau')$, *then* $\vdash C \approx^{log}_{\mathcal{E}\Rightarrow\mathcal{V}} C : (\Delta; \Gamma \vdash \theta) \Rightarrow (\Delta'; \Gamma' \vdash \tau')$.
3. *If* $\vdash C : (\Delta; \Gamma \vdash \tau) \Rightarrow (\Delta'; \Gamma' \vdash \theta')$, *then* $\vdash C \approx^{log}_{\mathcal{V}\Rightarrow\mathcal{E}} C : (\Delta; \Gamma \vdash \tau) \Rightarrow (\Delta'; \Gamma' \vdash \theta')$.
4. *If* $\vdash C : (\Delta; \Gamma \vdash \tau) \Rightarrow (\Delta'; \Gamma' \vdash \tau')$, *then* $\vdash C \approx^{log}_{\mathcal{V}\Rightarrow\mathcal{V}} C : (\Delta; \Gamma \vdash \tau) \Rightarrow (\Delta'; \Gamma' \vdash \tau')$.

Proof

By induction on the context typing derivation, applying appropriate compatibility at each step. □

7.2 Sound and Complete

Theorem 7.40 (Contextual Equivalence Implies CIU Equivalence)
If $\Delta; \Gamma \vdash e_1 \approx^{ctx}_{\mathrm{ST}} e_2 : \theta$, *then* $\Delta; \Gamma \vdash e_1 \approx^{ciu}_{\mathrm{ST}} e_2 : \theta$.

Proof

Since $\Delta; \Gamma \vdash e_1 \approx^{ctx}_{\mathrm{ST}} e_2 : \theta$, $\Delta; \Gamma \vdash e_1 : \theta$ and $\Delta; \Gamma \vdash e_2 : \theta$.

Suppose $\Delta \vDash \delta, \delta, \Gamma \vDash \gamma$ and $\vdash K : (\cdot; \cdot \vdash \theta) \Rightarrow (\cdot; \cdot \vdash \mathbf{1})$. We seek to prove that $K[\delta(\gamma(e_1))] \Updownarrow K[\delta(\gamma(e_2))]$.

First, we split Γ into $\mathsf{\Gamma} = \{(\mathsf{x}_1 : \sigma_1), \ldots, (\mathsf{x}_n : \sigma_n)\}$ and $\boldsymbol{\Gamma} = \{(\mathbf{x_1} : \boldsymbol{\tau_1}), \ldots, (\mathbf{x_m} : \boldsymbol{\tau_m})\}$ and define $\{\boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_p}\} = \Delta$.

For each x_i, define $\mathsf{C}_i = \mathsf{let}\, \mathsf{x}_i = \gamma(\mathsf{x}_i)\, \mathsf{in}\, [\cdot]$ and for each $\mathbf{x_i}$, define $\mathbf{C_i} = \mathbf{let}\, \mathbf{x_i} = \mathbf{return_0}\ \gamma(\mathbf{x_i})\, \mathbf{in}\, [\cdot]$. Next, for each $\boldsymbol{\alpha_i}$, define $\mathbf{C_{m+i}} = (\boldsymbol{\lambda}[\boldsymbol{\alpha_i}](\mathbf{y} : \mathbf{1}).\, [\cdot])\ [\delta(\boldsymbol{\alpha_i})]\ \langle\rangle$. Finally, define

$$\mathsf{C} = {}^{1}\mathcal{ST}\ \mathbf{C_{m+1}}[\ldots \mathbf{C_{m+p}}[\mathbf{C_1}[\ldots \mathbf{C_m}[\mathcal{TS}^{1}\ \mathsf{C}_1[\ldots \mathsf{C}_n[K]\ldots]]\ldots]]\ldots]$$

Then $\vdash \mathsf{C} : (\Delta; \Gamma \vdash \theta) \Rightarrow (\cdot; \cdot \vdash \mathsf{1})$, so since e_1, e_2 are contextually equivalent, $\mathsf{C}[e_1] \Updownarrow \mathsf{C}[e_2]$. Furthermore, $\mathsf{C}[e_1] \longmapsto^{p+m+n} {}^{1}\mathcal{ST}\,\mathcal{TS}^{1}\ K[\delta(\gamma(e_1))]$, so $\mathsf{C}[e_1] \Updownarrow {}^{1}\mathcal{ST}\,\mathcal{TS}^{1}\ K[\delta(\gamma(e_1))]$. Finally, by definition of the operational semantics, ${}^{1}\mathcal{ST}\,\mathcal{TS}^{1}\ K[\delta(\gamma(e_1))] \Updownarrow K[\delta(\gamma(e_1))]$, so $\mathsf{C}[e_1] \Updownarrow K[\delta(\gamma(e_1))]$. By analogous reasoning $\mathsf{C}[e_2] \Updownarrow K[\delta(\gamma(e_2))]$.

Therefore, by transitivity of $\Updownarrow$, $K[\delta(\gamma(e_1))] \Updownarrow K[\delta(\gamma(e_2))]$. □

Theorem 7.41 (CIU Equivalence Implies Logically Related)
If $\Delta; \Gamma \vdash e_1 \approx^{ciu}_{\mathrm{ST}} e_2 : \theta$, *then* $\Delta; \Gamma \vdash e_1 \approx^{log}_{\mathcal{E}} e_2 : \theta$.

Proof

Since $\Delta; \Gamma \vdash e_1 \approx^{ciu}_{\mathrm{ST}} e_2 : \theta$, $\Delta; \Gamma \vdash e_1 : \theta$ and $\Delta; \Gamma \vdash e_2 : \theta$.

Suppose $(k, K_1, K_2) \in \mathcal{K}\,[\![\theta]\!]\,\rho$, we seek to prove that $(k, K_1[\rho_1(\gamma_1(e_1))], K_2[\rho_2(\gamma_2(e_2))]) \in \mathcal{O}$.

Using $\Delta; \Gamma \vdash e_1 \approx^{ciu}_{\mathrm{ST}} e_2 : \theta$ twice and Theorem 7.38 twice, we get

1. $K_1[\rho_1(\gamma_1(e_1))] \Updownarrow K_1[\rho_1(\gamma_1(e_2))]$
2. $K_2[\rho_2(\gamma_2(e_1))] \Updownarrow K_2[\rho_2(\gamma_2(e_2))]$
3. $(k, K_1[\rho_1(\gamma_1(e_1))], K_2[\rho_2(\gamma_2(e_1))]) \in \mathcal{O}$
4. $(k, K_1[\rho_1(\gamma_1(e_2))], K_2[\rho_2(\gamma_2(e_2))]) \in \mathcal{O}$

By case analysis of 3:

Case $K_1[\rho_1(\gamma_1(e_1))] \Downarrow \wedge K_2[\rho_2(\gamma_2(e_1))] \Downarrow$: then by 2, $K_2[\rho_2(\gamma_2(e_2))] \Downarrow$.

Case $\mathrm{running}(k, K_1[\rho_1(\gamma_1(e_1))]) \wedge \mathrm{running}(k, K_2[\rho_2(\gamma_2(e_2))])$: By case analysis of 4:

Case $K_1[\rho_1(\gamma_1(e_2))] \Downarrow \wedge K_2[\rho_2(\gamma_2(e_2))] \Downarrow$: then by 1, $K_1[\rho_1(\gamma_1(e_1))] \Downarrow$.

Case $\text{running}(k, K_2[\rho_2(\gamma_2(e_2))]) \wedge \text{running}(k, K_1[\rho_1(\gamma_1(e_2))])$: then we have precisely that $\text{running}(k, K_1[\rho_1(\gamma_1(e_1))]) \wedge \text{running}(k, K_2[\rho_2(\gamma_2(e_2))])$.

$\square$

Theorem 7.42 (Logically Related Implies Contextual Equivalence)
If $\Delta; \Gamma \vdash e_1 \approx^{log}_{\mathcal{E}} e_2 : \theta$, *then* $\Delta; \Gamma \vdash e_1 \approx^{ctx}_{\mathrm{ST}} e_2 : \theta$.

Proof

Since $\Delta; \Gamma \vdash e_1 \approx^{log}_{\mathcal{E}} e_2 : \theta$, $\Delta; \Gamma \vdash e_1 : \theta$ and $\Delta; \Gamma \vdash e_2 : \theta$.

Suppose $\vdash C : (\Delta; \Gamma \vdash \theta) \Rightarrow (\cdot; \cdot \vdash 1)$. Then by Lemma 7.39, $\cdot; \cdot \vdash C[e_1] \approx^{log}_{\mathcal{E}} C[e_2] : 1$.

We seek to prove that $C[e_1] \Updownarrow C[e_2]$. Suppose $C[e_1] \Downarrow$. Then in particular there exists some $k \geq 0$ such that $\neg\text{running}(C[e_1], k)$. Furthermore, since $\cdot; \cdot \vdash C[e_1] \approx^{log}_{\mathcal{E}} C[e_2] : 1$, $(k, C[e_1], C[e_2]) \in \mathcal{O}$, so since $\neg\text{running}(C[e_1], k)$, $C[e_2] \Downarrow$. By symmetric reasoning, if $C[e_2] \Downarrow$, then $C[e_1] \Downarrow$. $\square$

Theorem 7.43 (Logical Relation, Contextual Equivalence, CIU Equivalence Coincide)
$\Delta; \Gamma \vdash e \approx^{log}_{\mathcal{E}} e' : \theta$ *if and only if* $\Delta; \Gamma \vdash e \approx^{ctx}_{\mathrm{ST}} e' : \theta$ *if and only if* $\Delta; \Gamma \vdash e \approx^{ciu}_{\mathrm{ST}} e' : \theta$

Proof

By Lemma 7.40, Lemma 7.41, and Lemma 7.42. $\square$

Theorem 7.44 (Logical Relation is Transitive)
If $\Delta; \Gamma \vdash e \approx^{log}_{\mathcal{E}} e' : \theta$ *and* $\Delta; \Gamma \vdash e' \approx^{log}_{\mathcal{E}} e'' : \theta$, *then* $\Delta; \Gamma \vdash e \approx^{log}_{\mathcal{E}} e'' : \theta$.

Proof

By Theorem 7.43 and transitivity of contextual equivalence. $\square$

8 Back-Translation From λ^{ST} to λ^{S}

$$
\begin{array}{lcl}
\delta & ::= & \emptyset \mid \delta[\alpha \mapsto \sigma, x] \\
\emptyset_{\Gamma} & \stackrel{\text{def}}{=} & \cdot \\
(\delta[\alpha \mapsto \sigma, x])_{\Gamma} & \stackrel{\text{def}}{=} & \delta_{\Gamma}, x : 1 \to ((\sigma \to U) \times (U \to \sigma)) \\
\emptyset_{\sigma} & \stackrel{\text{def}}{=} & \emptyset \\
(\delta[\alpha \mapsto \sigma, x])_{\sigma} & \stackrel{\text{def}}{=} & \delta_{\sigma}[\alpha \mapsto \sigma] \\
\emptyset_{x} & \stackrel{\text{def}}{=} & \emptyset \\
(\delta[\alpha \mapsto \sigma, x])_{x} & \stackrel{\text{def}}{=} & \delta_{x}[\alpha \mapsto x]
\end{array}
$$

Figure 21: Embedding-Projection Environment

$$
\begin{array}{lcl}
U & \stackrel{\text{def}}{=} & \mu\alpha.\, 1 + (\alpha + \alpha) + (\alpha \times \alpha) + (\alpha \to R(\alpha)) + \alpha \\
R(\sigma) & \stackrel{\text{def}}{=} & \sigma + \sigma \\
R & \stackrel{\text{def}}{=} & R(U)
\end{array}
$$

Figure 22: Universal Type and Result Type

$$\boxed{\text{if } \Delta;\Gamma \vdash \mathsf{v}_f : (\sigma_1 \to \sigma_2) \to \sigma_1 \to \sigma_2, \text{ then } \Delta;\Gamma \vdash \underline{\mathsf{FIX}}_{\sigma_1\to\sigma_2}(\mathsf{v}_f) : \sigma_1 \to \sigma_2}$$

$$\boxed{\text{if } \Delta;\Gamma \vdash \mathsf{v}_f : (\sigma_1 \to \sigma_2) \to \sigma_1 \to \sigma_2, \text{ then } \Delta;\Gamma \vdash \underline{\mathsf{LOOP}}(\sigma_1 \to \sigma_2, \mathsf{v}_f) : (\mu\alpha.\,\alpha \to \sigma_1 \to \sigma_2) \to \sigma_1 \to \sigma_2}$$

$$\begin{array}{lcl}
\underline{\mathsf{FIX}}_{\sigma_1\to\sigma_2}(\mathsf{v}_f) & \stackrel{\text{def}}{=} & \lambda(\mathsf{z} : \sigma_1). \\
& & \quad \mathsf{let}\ \mathsf{x}_{fix} = \underline{\mathsf{LOOP}}(\sigma_1 \to \sigma_2, \mathsf{v}_f)\ (\mathsf{fold}_{\mu\alpha.\alpha\to\sigma_1\to\sigma_2}\ \underline{\mathsf{LOOP}}(\sigma_1 \to \sigma_2, \mathsf{v}_f))\ \mathsf{in} \\
& & \quad \mathsf{let}\ \mathsf{x}_f = \mathsf{v}_f\ \mathsf{x}_{fix}\ \mathsf{in} \\
& & \quad \mathsf{x}_f\ \mathsf{z} \\
\underline{\mathsf{LOOP}}(\sigma_1 \to \sigma_2, \mathsf{v}_f) & \stackrel{\text{def}}{=} & \lambda(\mathsf{x}_{folded} : \mu\alpha.\,\alpha \to \sigma_1 \to \sigma_2). \\
& & \quad \mathsf{let}\ \mathsf{x}_{loop} = \mathsf{unfold}\ \mathsf{x}_{folded}\ \mathsf{in} \\
& & \quad \lambda(\mathsf{z} : \sigma_1). \\
& & \quad\quad \mathsf{let}\ \mathsf{x}_{fix} = \mathsf{x}_{loop}\ (\mathsf{fold}_{\mu\alpha.\alpha\to\sigma_1\to\sigma_2}\ \mathsf{x}_{loop})\ \mathsf{in} \\
& & \quad\quad \mathsf{let}\ \mathsf{x}_f = \mathsf{v}_f\ \mathsf{x}_{fix}\ \mathsf{in} \\
& & \quad\quad \mathsf{x}_f\ \mathsf{z} \\
\underline{\mathsf{UNIT}} & \stackrel{\text{def}}{=} & \mathsf{fold}_{\mathsf{U}}\ (\mathsf{inj}_1\ \langle\rangle) \\
\underline{\mathsf{IN}}(\mathsf{i}, \mathsf{v}) & \stackrel{\text{def}}{=} & \mathsf{fold}_{\mathsf{U}}\ (\mathsf{inj}_2\ (\mathsf{inj}_1\ (\mathsf{inj}_{\mathsf{i}}\ \mathsf{v}))) \\
\underline{\mathsf{CONS}}(\mathsf{v}_1, \mathsf{v}_2) & \stackrel{\text{def}}{=} & \mathsf{fold}_{\mathsf{U}}\ (\mathsf{inj}_2\ (\mathsf{inj}_2\ (\mathsf{inj}_1\ \langle \mathsf{v}_1, \mathsf{v}_2\rangle))) \\
\underline{\mathsf{LAMBDA}}(\lambda(\mathsf{x} : \mathsf{U}).\,\mathsf{e}) & \stackrel{\text{def}}{=} & \mathsf{fold}_{\mathsf{U}}\ (\mathsf{inj}_2\ (\mathsf{inj}_2\ (\mathsf{inj}_2\ (\mathsf{inj}_1\ (\lambda(\mathsf{x} : \mathsf{U}).\,\mathsf{e}))))) \\
\underline{\mathsf{FOLD}}(\mathsf{v}) & \stackrel{\text{def}}{=} & \mathsf{fold}_{\mathsf{U}}\ (\mathsf{inj}_2\ (\mathsf{inj}_2\ (\mathsf{inj}_2\ (\mathsf{inj}_2\ (\mathsf{v}))))) \\
\underline{\mathsf{RETURN}}(\mathsf{v}) & \stackrel{\text{def}}{=} & \mathsf{inj}_1\ \mathsf{v} \\
\underline{\mathsf{RAISE}}(\mathsf{v}) & \stackrel{\text{def}}{=} & \mathsf{inj}_2\ \mathsf{v} \\
\underline{\mathsf{TOLHS}}(\mathsf{v}_u) & \stackrel{\text{def}}{=} & \mathsf{case}\ \mathsf{v}_u\ \mathsf{of}\ \mathsf{x}_1.\,\mathsf{x}_1 \mid \mathsf{x}_2.\,\mho \\
\underline{\mathsf{TORHS}}(\mathsf{v}_u) & \stackrel{\text{def}}{=} & \mathsf{case}\ \mathsf{v}_u\ \mathsf{of}\ \mathsf{x}_1.\,\mho \mid \mathsf{x}_2.\,\mathsf{x}_2 \\
\underline{\mathsf{TOSUM}}(\mathsf{v}_u) & \stackrel{\text{def}}{=} & \mathsf{let}\ \mathsf{x}_1 = \mathsf{unfold}\ \mathsf{v}_u\ \mathsf{in} \\
& & \mathsf{let}\ \mathsf{x}_2 = \underline{\mathsf{TORHS}}(\mathsf{x}_1)\ \mathsf{in} \\
& & \underline{\mathsf{TOLHS}}(\mathsf{x}_2) \\
\underline{\mathsf{TOPAIR}}(\mathsf{v}_u) & \stackrel{\text{def}}{=} & \mathsf{let}\ \mathsf{x}_1 = \mathsf{unfold}\ \mathsf{v}_u\ \mathsf{in} \\
& & \mathsf{let}\ \mathsf{x}_2 = \underline{\mathsf{TORHS}}(\mathsf{x}_1)\ \mathsf{in} \\
& & \mathsf{let}\ \mathsf{x}_3 = \underline{\mathsf{TORHS}}(\mathsf{x}_2)\ \mathsf{in}\ \underline{\mathsf{TOLHS}}(\mathsf{x}_3) \\
\underline{\mathsf{TOFUN}}(\mathsf{v}_u) & \stackrel{\text{def}}{=} & \mathsf{let}\ \mathsf{x}_1 = \mathsf{unfold}\ \mathsf{v}_u\ \mathsf{in} \\
& & \mathsf{let}\ \mathsf{x}_2 = \underline{\mathsf{TORHS}}(\mathsf{x}_1)\ \mathsf{in} \\
& & \mathsf{let}\ \mathsf{x}_3 = \underline{\mathsf{TORHS}}(\mathsf{x}_2)\ \mathsf{in} \\
& & \mathsf{let}\ \mathsf{x}_4 = \underline{\mathsf{TORHS}}(\mathsf{x}_3)\ \mathsf{in}\ \underline{\mathsf{TOLHS}}(\mathsf{x}_4) \\
\underline{\mathsf{TOFOLD}}(\mathsf{v}_u) & \stackrel{\text{def}}{=} & \mathsf{let}\ \mathsf{x}_1 = \mathsf{unfold}\ \mathsf{v}_u\ \mathsf{in} \\
& & \mathsf{let}\ \mathsf{x}_2 = \underline{\mathsf{TORHS}}(\mathsf{x}_1)\ \mathsf{in} \\
& & \mathsf{let}\ \mathsf{x}_3 = \underline{\mathsf{TORHS}}(\mathsf{x}_2)\ \mathsf{in} \\
& & \mathsf{let}\ \mathsf{x}_4 = \underline{\mathsf{TORHS}}(\mathsf{x}_3)\ \mathsf{in}\ \underline{\mathsf{TORHS}}(\mathsf{x}_4) \\
\underline{\mathsf{PRJ}}(1, \mathsf{v}_u) & \stackrel{\text{def}}{=} & \mathsf{let}\ \mathsf{x} = \underline{\mathsf{TOPAIR}}(\mathsf{v}_u)\ \mathsf{in}\ \pi_1\,\mathsf{x} \\
\underline{\mathsf{PRJ}}(\mathsf{i}+1, \mathsf{v}_u) & \stackrel{\text{def}}{=} & \mathsf{let}\ \mathsf{x} = \underline{\mathsf{TOPAIR}}(\mathsf{v}_u)\ \mathsf{in} \\
& & \mathsf{let}\ \mathsf{y} = \pi_2\,\mathsf{x}\ \mathsf{in}\ \underline{\mathsf{PRJ}}(\mathsf{i}, \mathsf{x})
\end{array}$$

Figure 23: Interpreter Metafunctions

$\boxed{\emptyset \vdash \underline{\mathsf{PROJECT}}(\sigma) : \mathsf{R} \to \sigma}$

$\boxed{\delta_\Gamma \vdash \underline{\mathsf{PROJECT}}(\delta, \sigma) : \mathsf{U} \to \delta_\sigma(\sigma)}$

$$
\begin{array}{lcl}
\underline{\mathsf{PROJECT}}(\sigma) & \stackrel{\text{def}}{=} & \lambda(\mathsf{x}_r : \mathsf{R}).\ \mathsf{let}\ \mathsf{x}_u = \underline{\mathsf{TOLHS}}(\mathsf{x}_r)\ \mathsf{in}\ \underline{\mathsf{PROJECT}}(\emptyset, \sigma)\ \mathsf{x}_u \\
\underline{\mathsf{PROJECT}}(\delta, \alpha) & \stackrel{\text{def}}{=} & \lambda(\mathsf{x}_u : \mathsf{U}).\ \mathsf{let}\ \mathsf{x} = \delta_{\mathsf{x}}(\alpha)\ \langle\rangle\ \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\ \mathsf{x}_f = \pi_2\, \mathsf{x}\ \mathsf{in}\ \mathsf{x}'\ \mathsf{x} \\
\underline{\mathsf{PROJECT}}(\delta, 1) & \stackrel{\text{def}}{=} & \lambda(\mathsf{x}_u : \mathsf{U}).\ \langle\rangle \\
\underline{\mathsf{PROJECT}}(\delta, \sigma_1 + \sigma_2) & \stackrel{\text{def}}{=} & \lambda(\mathsf{x}_u : \mathsf{U}).\ \mathsf{let}\ \mathsf{x} = \underline{\mathsf{TOSUM}}(\mathsf{x}_u)\ \mathsf{in} \\
& & \qquad\qquad \mathsf{case}\ \mathsf{x}\ \mathsf{of} \\
& & \qquad\qquad\ \mathsf{x}_1 \,.\ \mathsf{let}\ \mathsf{x}_1' = \underline{\mathsf{PROJECT}}(\delta, \sigma_1)\ \mathsf{x}_1\ \mathsf{in}\ \mathsf{inj}_1\, \mathsf{x}_1' \\
& & \qquad\qquad\ \mathsf{x}_2 \,.\ \mathsf{let}\ \mathsf{x}_2' = \underline{\mathsf{PROJECT}}(\delta, \sigma_2)\ \mathsf{x}_2\ \mathsf{in}\ \mathsf{inj}_2\, \mathsf{x}_2' \\
\underline{\mathsf{PROJECT}}(\delta, \sigma_1 \times \sigma_2) & \stackrel{\text{def}}{=} & \lambda(\mathsf{x}_u : \mathsf{U}).\ \mathsf{let}\ \mathsf{x} = \underline{\mathsf{TOPAIR}}(\mathsf{x}_u)\ \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\ \mathsf{x}_1 = \pi_1\, \mathsf{x}\ \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\ \mathsf{x}_1' = \underline{\mathsf{PROJECT}}(\delta, \sigma_1)\ \mathsf{x}_1\ \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\ \mathsf{y} = \pi_2\, \mathsf{x}\ \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\ \mathsf{y}' = \underline{\mathsf{TOPAIR}}(\mathsf{y})\ \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\ \mathsf{x}_2 = \pi_1\, \mathsf{y}'\ \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\ \mathsf{x}_2' = \underline{\mathsf{PROJECT}}(\delta, \sigma_2)\ \mathsf{x}_2\ \mathsf{in} \\
& & \qquad\qquad \langle \mathsf{x}_1', \mathsf{x}_2' \rangle \\
\underline{\mathsf{PROJECT}}(\delta, \sigma_1 \to \sigma_2) & \stackrel{\text{def}}{=} & \lambda(\mathsf{x}_u : \mathsf{U}).\ \mathsf{let}\ \mathsf{x}_u' = \underline{\mathsf{TOPAIR}}(\mathsf{x}_u)\ \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\ \mathsf{x}_f = \underline{\mathsf{PRJ}}(1, \mathsf{x}_u')\ \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\ \mathsf{x}_{env} = \underline{\mathsf{PRJ}}(2, \mathsf{x}_u')\ \mathsf{in} \\
& & \qquad\qquad \lambda(\mathsf{y} : \delta_\sigma(\sigma_1)).\ \mathsf{let}\ \mathsf{y}_u = \underline{\mathsf{EMBED}}(\delta, \sigma_1)\ \mathsf{y}\ \mathsf{in} \\
& & \qquad\qquad\qquad\qquad\qquad \mathsf{let}\ \mathsf{x} = \underline{\mathsf{CONS}}(\mathsf{x}_{env}, \underline{\mathsf{CONS}}(\mathsf{y}_u, \underline{\mathsf{UNIT}}))\ \mathsf{in} \\
& & \qquad\qquad\qquad\qquad\qquad \mathsf{let}\ \mathsf{x}_r = \mathsf{x}_f\ \mathsf{x}\ \mathsf{in} \\
& & \qquad\qquad\qquad\qquad\qquad \mathsf{let}\ \mathsf{x}_u'' = \underline{\mathsf{TOLHS}}(\mathsf{x}_r)\ \mathsf{in} \\
& & \qquad\qquad\qquad\qquad\qquad \underline{\mathsf{PROJECT}}(\delta, \sigma_2)\ \mathsf{x}_u'' \\
\underline{\mathsf{PROJECT}}(\delta, \mu\alpha.\, \sigma) & \stackrel{\text{def}}{=} & \lambda(\mathsf{x}_u : \mathsf{U}).\ \mathsf{let}\ \mathsf{x} = \underline{\mathsf{EP}}(\delta, \mu\alpha.\, \sigma)\ \langle\rangle\ \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\ \mathsf{x}_f = \pi_2\, \mathsf{x}\ \mathsf{in}\ \mathsf{x}_f\ \mathsf{x}_u
\end{array}
$$

Figure 24: Projecting from the Universal Type

$\boxed{\emptyset \vdash \underline{\mathsf{EMBED}}(\sigma) : \sigma \to \mathsf{R}}$

$\boxed{\delta_\Gamma \vdash \underline{\mathsf{EMBED}}(\delta, \sigma) : \delta_\sigma(\sigma) \to \mathsf{U}}$

$$
\begin{array}{lcl}
\underline{\mathsf{EMBED}}(\sigma) & \stackrel{\mathrm{def}}{=} & \lambda(\mathsf{x} : \sigma).\, \mathsf{let}\, \mathsf{x}_u = \underline{\mathsf{EMBED}}(\emptyset, \sigma)\ \mathsf{x}\, \mathsf{in}\, \underline{\mathsf{RETURN}}(\mathsf{x}_u) \\
\underline{\mathsf{EMBED}}(\delta, \alpha) & \stackrel{\mathrm{def}}{=} & \lambda(\mathsf{x} : \delta_\sigma(\alpha)).\, \mathsf{let}\, \mathsf{x}_{ep} = \delta_{\mathsf{x}}(\alpha)\ \langle\rangle\ \mathsf{in} \\
& & \qquad \mathsf{let}\, \mathsf{x}_{embed} = \pi_1\, \mathsf{x}_{ep}\, \mathsf{in}\, \mathsf{x}_{embed}\ \mathsf{x} \\
\underline{\mathsf{EMBED}}(\delta, 1) & \stackrel{\mathrm{def}}{=} & \lambda(\mathsf{x} : \delta_\sigma(1)).\, \underline{\mathsf{UNIT}} \\
\underline{\mathsf{EMBED}}(\delta, \sigma_1 + \sigma_2) & \stackrel{\mathrm{def}}{=} & \lambda(\mathsf{x} : \delta_\sigma(\sigma_1 + \sigma_2)).\, \mathsf{case}\, \mathsf{x}\, \mathsf{of} \\
& & \qquad \mathsf{x}_1 .\, \mathsf{let}\, \mathsf{x}' = \underline{\mathsf{EMBED}}(\delta, \sigma_1)\ \mathsf{x}_1\ \mathsf{in} \\
& & \qquad\quad \underline{\mathsf{IN}}(1, \mathsf{x}') \\
& & \qquad \mathsf{x}_2 .\, \mathsf{let}\, \mathsf{x}' = \underline{\mathsf{EMBED}}(\delta, \sigma_2)\ \mathsf{x}_2\ \mathsf{in} \\
& & \qquad\quad \underline{\mathsf{IN}}(2, \mathsf{x}') \\
\underline{\mathsf{EMBED}}(\delta, \sigma_1 \times \sigma_2) & \stackrel{\mathrm{def}}{=} & \lambda(\mathsf{x} : \delta_\sigma(\sigma_1 \times \sigma_2)).\, \mathsf{let}\, \mathsf{x}_1 = \pi_1\, \mathsf{x}\, \mathsf{in} \\
& & \qquad \mathsf{let}\, \mathsf{x}_2 = \pi_2\, \mathsf{x}\, \mathsf{in} \\
& & \qquad \mathsf{let}\, \mathsf{x}_1' = \underline{\mathsf{EMBED}}(\delta, \sigma_1)\ \mathsf{x}_1\ \mathsf{in} \\
& & \qquad \mathsf{let}\, \mathsf{x}_2' = \underline{\mathsf{EMBED}}(\delta, \sigma_2)\ \mathsf{x}_2\ \mathsf{in} \\
& & \qquad\quad \underline{\mathsf{CONS}}(\mathsf{x}_1', \underline{\mathsf{CONS}}(\mathsf{x}_2', \underline{\mathsf{UNIT}})) \\
\underline{\mathsf{EMBED}}(\delta, \sigma_1 \to \sigma_2) & \stackrel{\mathrm{def}}{=} & \lambda(\mathsf{x}_f : \delta_\sigma(\sigma_1 \to \sigma_2)).\, \mathsf{let}\, \mathsf{x}_f' = \lambda(\mathsf{x}_u : \mathsf{U}).\, \mathsf{let}\, \mathsf{x}_u' = \underline{\mathsf{PRJ}}(2, \mathsf{x}_u)\ \mathsf{in} \qquad \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\, \mathsf{x} = \underline{\mathsf{PROJECT}}(\delta, \sigma_1)\ \mathsf{x}_u'\ \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\, \mathsf{y} = \mathsf{x}_f\ \mathsf{x}\, \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\, \mathsf{x}_u'' = \underline{\mathsf{EMBED}}(\delta, \sigma_2)\ \mathsf{y}\ \mathsf{in} \\
& & \qquad\qquad \underline{\mathsf{RETURN}}(\mathsf{x}_u'') \\
& & \qquad \underline{\mathsf{CONS}}(\mathsf{x}_f', \underline{\mathsf{CONS}}(\underline{\mathsf{UNIT}}, \underline{\mathsf{UNIT}})) \\
\underline{\mathsf{EMBED}}(\delta, \mu\alpha.\, \sigma) & \stackrel{\mathrm{def}}{=} & \lambda(\mathsf{x} : \delta_\sigma(\mu\alpha.\, \sigma)).\, \mathsf{let}\, \mathsf{x}_{ep} = \underline{\mathsf{EP}}(\delta, \mu\alpha.\, \sigma)\ \langle\rangle\ \mathsf{in} \\
& & \qquad \mathsf{let}\, \mathsf{x}_{embed} = \pi_1\, \mathsf{x}_{ep}\, \mathsf{in}\, \mathsf{x}_{embed}\ \mathsf{x}
\end{array}
$$

Figure 25: Embedding into the Universal Type

$\boxed{\delta_\Gamma \vdash \underline{\mathsf{EP}}(\delta, \mu\alpha.\, \sigma) : 1 \to ((\delta_\sigma(\mu\alpha.\, \sigma) \to \mathsf{U}) \times (\mathsf{U} \to \delta_\sigma(\mu\alpha.\, \sigma)))}$

$$
\begin{array}{lcl}
\underline{\mathsf{EP}}(\delta, \mu\alpha.\, \sigma) & \stackrel{\mathrm{def}}{=} & \underline{\mathsf{FIX}}_{1 \to ((\delta_\sigma(\mu\alpha.\sigma) \to \mathsf{U}) \times (\mathsf{U} \to \delta_\sigma(\mu\alpha.\sigma)))} \\
& & \lambda(\mathsf{x}_{\mu\alpha.\sigma} : 1 \to ((\delta_\sigma(\mu\alpha.\, \sigma) \to \mathsf{U}) \times (\mathsf{U} \to \delta_\sigma(\mu\alpha.\, \sigma)))). \\
& & \quad \lambda(\mathsf{x}_{unit} : 1). \\
& & \qquad \mathsf{let}\, \mathsf{x}_{embed} = \\
& & \qquad\ \lambda(\mathsf{x} : \delta_\sigma(\mu\alpha.\, \sigma)). \\
& & \qquad\quad \mathsf{let}\, \mathsf{y} = \mathsf{unfold}\, \mathsf{x}\, \mathsf{in} \\
& & \qquad\quad \mathsf{let}\, \mathsf{y}_u = \underline{\mathsf{EMBED}}(\delta[\alpha \mapsto \mu\alpha.\, \sigma, \mathsf{x}_{\mu\alpha.\sigma}], \sigma)\ \mathsf{y}\, \mathsf{in} \\
& & \qquad\quad \underline{\mathsf{FOLD}}(\mathsf{y}_u) \\
& & \qquad \mathsf{in}\, \mathsf{let}\, \mathsf{x}_{project} = \\
& & \qquad\quad \lambda(\mathsf{x}_u : \mathsf{U}). \\
& & \qquad\qquad \mathsf{let}\, \mathsf{y}_u = \underline{\mathsf{TOFOLD}}(\mathsf{x}_u)\ \mathsf{in} \\
& & \qquad\qquad \mathsf{let}\, \mathsf{y} = \underline{\mathsf{PROJECT}}(\delta[\alpha \mapsto \mu\alpha.\, \sigma, \mathsf{x}_{\mu\alpha.\sigma}], \sigma)\ \mathsf{y}_u\ \mathsf{in} \\
& & \qquad\qquad \mathsf{fold}_{\mu\alpha.\sigma}\, \mathsf{y} \\
& & \qquad\quad \mathsf{in}\, \langle \mathsf{x}_{embed}, \mathsf{x}_{project} \rangle
\end{array}
$$

Figure 26: Embedding-Projection Pair for Recursive Types

$\boxed{^{\twoheadrightarrow} : \Gamma \to \Gamma}$

$$\begin{array}{lcl}
(\cdot)^{\twoheadrightarrow} & = & \cdot \\
(\Gamma, x : \sigma)^{\twoheadrightarrow} & = & \Gamma^{\twoheadrightarrow}, x : \sigma \\
(\Gamma, \mathbf{y} : \boldsymbol{\tau})^{\twoheadrightarrow} & = & \Gamma^{\twoheadrightarrow}, y : U
\end{array}$$

$\boxed{\Delta; \Gamma \vdash e : \sigma \twoheadrightarrow e'}$

where $e' \in \lambda^{S}$ and $\Delta; \Gamma \vdash e : \sigma$ and $\Gamma^{\twoheadrightarrow} \vdash e' : \sigma$

$$\frac{}{\Delta; \Gamma \vdash x : \sigma \twoheadrightarrow x} \qquad \frac{\Delta; \Gamma \vdash v : \sigma_i \twoheadrightarrow v' \qquad i \in \{1, 2\}}{\Delta; \Gamma \vdash \mathsf{inj}_i\, v : \sigma_1 + \sigma_2 \twoheadrightarrow \mathsf{inj}_i\, v'}$$

$$\frac{\Delta; \Gamma \vdash v : \sigma_1 + \sigma_2 \twoheadrightarrow v' \qquad \Delta; \Gamma, x_1 : \sigma_1 \vdash e_1 : \sigma \twoheadrightarrow e_1' \qquad \Delta; \Gamma, x_2 : \sigma_2 \vdash e_2 : \sigma \twoheadrightarrow e_2'}{\Delta; \Gamma \vdash \mathsf{case}\, v\, \mathsf{of}\, x_1.\, e_1 \mid x_2.\, e_2 : \sigma \twoheadrightarrow \mathsf{case}\, v'\, \mathsf{of}\, x_1.\, e_1' \mid x_2.\, e_2'}$$

$$\frac{\Delta; \Gamma \vdash v_1 : \sigma_1 \twoheadrightarrow v_1' \qquad \Delta; \Gamma \vdash v_2 : \sigma_2 \twoheadrightarrow v_2'}{\Delta; \Gamma \vdash \langle v_1, v_2 \rangle : \sigma_1 \times \sigma_2 \twoheadrightarrow \langle v_1', v_2' \rangle} \qquad \frac{\Delta; \Gamma \vdash v : \sigma_1 \times \sigma_2 \twoheadrightarrow v' \qquad i \in \{1, 2\}}{\Delta; \Gamma \vdash \pi_i\, v : \sigma_i \twoheadrightarrow \pi_i\, v'}$$

$$\frac{\Delta; \Gamma, x : \sigma_1 \vdash e : \sigma_2 \twoheadrightarrow e'}{\Delta; \Gamma \vdash \lambda(x : \sigma_1).\, e : \sigma_1 \to \sigma_2 \twoheadrightarrow \lambda(x : \sigma_1).\, e'} \qquad \frac{\Delta; \Gamma \vdash v_1 : \sigma_2 \to \sigma \twoheadrightarrow v_1' \qquad \Delta; \Gamma \vdash v_2 : \sigma_2 \twoheadrightarrow v_2'}{\Delta; \Gamma \vdash v_1\ v_2 : \sigma \twoheadrightarrow v_1'\ v_2'}$$

$$\frac{\Delta; \Gamma \vdash v : \sigma[\mu\alpha.\, \sigma/\alpha] \twoheadrightarrow v'}{\Delta; \Gamma \vdash \mathsf{fold}_{\mu\alpha.\sigma}\, v : \mu\alpha.\, \sigma \twoheadrightarrow \mathsf{fold}_{\mu\alpha.\sigma}\, v'} \qquad \frac{\Delta; \Gamma \vdash v : \mu\alpha.\, \sigma \twoheadrightarrow v'}{\Delta; \Gamma \vdash \mathsf{unfold}\, v : \sigma[\mu\alpha.\, \sigma/\alpha] \twoheadrightarrow \mathsf{unfold}\, v'}$$

$$\frac{\Delta; \Gamma \vdash e_1 : \sigma_1 \twoheadrightarrow e_1' \qquad \Delta; \Gamma, x : \sigma_1 \vdash e_2 : \sigma_2 \twoheadrightarrow e_2'}{\Delta; \Gamma \vdash \mathsf{let}\, x = e_1\, \mathsf{in}\, e_2 : \sigma_2 \twoheadrightarrow \mathsf{let}\, x = e_1'\, \mathsf{in}\, e_2'} \qquad \frac{\Delta; \Gamma \vdash^{\div} \mathbf{e} : \sigma^{\div} \twoheadrightarrow e_u}{\Delta; \Gamma \vdash {}^{\sigma}\mathcal{ST}\, \mathbf{e} : \sigma \twoheadrightarrow \mathsf{let}\, x = e_u\, \mathsf{in}\, \underline{\mathsf{PROJECT}}(\sigma)\ x}$$

Figure 27: Relating λ^{ST} terms to λ^{S} terms ("Back-Translation")

$\boxed{\Delta;\Gamma \vdash^{+} \mathbf{v} : \boldsymbol{\tau} \twoheadrightarrow \mathsf{v}}$ where $\mathsf{v} \in \lambda^{S}$ and $\Delta;\Gamma \vdash \mathbf{v} : \boldsymbol{\tau}$ and $\Gamma^{\twoheadrightarrow} \vdash \mathsf{v} : \mathsf{U}$

$$\frac{}{\Delta;\Gamma \vdash^{+} \mathbf{y} : \sigma^{+} \twoheadrightarrow \mathsf{y}} \qquad \frac{}{\Delta;\Gamma \vdash^{+} \langle\rangle : \langle\rangle \twoheadrightarrow \underline{\mathsf{UNIT}}} \qquad \frac{\Delta;\Gamma \vdash^{+} \mathbf{v} : \boldsymbol{\tau}_{\mathbf{i}} \twoheadrightarrow \mathsf{v}_u}{\Delta;\Gamma \vdash^{+} \mathbf{inj_i}\, \mathbf{v} : \boldsymbol{\tau_1} + \boldsymbol{\tau_2} \twoheadrightarrow \underline{\mathsf{IN}}(\mathsf{i}, \mathsf{v}_u)}$$

$$\frac{\Delta;\Gamma \vdash^{+} \mathbf{v_1} : \boldsymbol{\tau} \twoheadrightarrow \mathsf{v} \qquad \Delta;\Gamma \vdash^{+} \langle \mathbf{v_1}, \ldots, \mathbf{v_n} \rangle : \langle \boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_n} \rangle \twoheadrightarrow \mathsf{v}'}{\Delta;\Gamma \vdash^{+} \langle \mathbf{v}, \mathbf{v_1}, \ldots, \mathbf{v_n} \rangle : \langle \boldsymbol{\tau}, \boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_n} \rangle \twoheadrightarrow \underline{\mathsf{CONS}}(\mathsf{v}, \mathsf{v}')}$$

$$\frac{\boldsymbol{\alpha}; \mathbf{x} : \boldsymbol{\tau} \vdash^{\div} \mathbf{e} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{e}_u}{\Delta;\Gamma \vdash^{+} \boldsymbol{\lambda}[\boldsymbol{\alpha}](\mathbf{x} : \boldsymbol{\tau}).\, \mathbf{e} : \forall[\boldsymbol{\alpha}].\, \boldsymbol{\tau} \to \boldsymbol{\theta} \twoheadrightarrow \underline{\mathsf{LAMBDA}}(\lambda(\mathsf{x} : \mathsf{U}).\, \mathsf{e}_u)} \qquad \frac{\Delta;\Gamma \vdash^{+} \mathbf{v} : \boldsymbol{\tau}[\boldsymbol{\mu\alpha.\tau}/\boldsymbol{\alpha}] \twoheadrightarrow \mathsf{v}_u}{\Delta;\Gamma \vdash^{+} \mathbf{fold}_{\boldsymbol{\mu\alpha.\tau}}\, \mathbf{v} : \boldsymbol{\mu\alpha.\tau} \twoheadrightarrow \underline{\mathsf{FOLD}}(\mathsf{v}_u)}$$

$$\frac{\Delta;\Gamma \vdash^{+} \mathbf{v} : \boldsymbol{\tau}[\boldsymbol{\tau}'/\boldsymbol{\alpha}] \twoheadrightarrow \mathsf{v}_u}{\Delta;\Gamma \vdash^{+} \mathbf{pack}\,(\boldsymbol{\tau}', \mathbf{v})\, \mathbf{as}\, \exists\boldsymbol{\alpha}.\, \boldsymbol{\tau} : \exists\boldsymbol{\alpha}.\, \boldsymbol{\tau} \twoheadrightarrow \mathsf{v}_u}$$

$\boxed{\Delta;\Gamma \vdash^{\div} \mathbf{r} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{v}_u}$ where $\mathsf{e} \in \lambda^{S}$ and $\Delta;\Gamma \vdash \mathbf{r} : \boldsymbol{\theta}$ and $\Gamma^{\twoheadrightarrow} \vdash \mathsf{v}_u : \mathsf{R}$

$$\frac{\Delta;\Gamma \vdash^{+} \mathbf{v} : \boldsymbol{\tau} \twoheadrightarrow \mathsf{v}_u}{\Delta;\Gamma \vdash^{\div} \mathbf{return}\ \mathbf{v} : \mathbf{E}\, \boldsymbol{\tau}_{\mathbf{exn}}\, \boldsymbol{\tau} \twoheadrightarrow \underline{\mathsf{RETURN}}(\mathsf{v}_u)} \qquad \frac{\Delta;\Gamma \vdash^{+} \mathbf{v} : \boldsymbol{\tau}_{\mathbf{exn}} \twoheadrightarrow \mathsf{v}_u}{\Delta;\Gamma \vdash^{\div} \mathbf{raise}\ \mathbf{v} : \mathbf{E}\, \boldsymbol{\tau}_{\mathbf{exn}}\, \boldsymbol{\tau} \twoheadrightarrow \underline{\mathsf{RAISE}}(\mathsf{v}_u)}$$

$\boxed{\Delta;\Gamma \vdash^{\div} \mathbf{e} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{e}}$ where $\mathsf{e} \in \lambda^{S}$ and $\Delta;\Gamma \vdash \mathbf{e} : \boldsymbol{\theta}$ and $\Gamma^{\twoheadrightarrow} \vdash \mathsf{e} : \mathsf{R}$

$$\frac{\Delta;\Gamma \vdash \mathsf{e} : \sigma \twoheadrightarrow \mathsf{e}'}{\Delta;\Gamma \vdash^{\div} \mathcal{TS}^{\sigma}\, \mathsf{e} : \sigma^{\div} \twoheadrightarrow \mathsf{let}\, \mathsf{x} = \mathsf{e}'\, \mathsf{in}\, \underline{\mathsf{EMBED}}(\sigma)\ \mathsf{x}}$$

$$\frac{\Delta;\Gamma \vdash^{+} \mathbf{v} : \boldsymbol{\tau_1} + \boldsymbol{\tau_2} \twoheadrightarrow \mathsf{v}_u \qquad \Delta;\Gamma, \mathbf{x_1} : \boldsymbol{\tau_1} \vdash^{\div} \mathbf{e_1} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{e}_1 \qquad \Delta;\Gamma, \mathbf{x_2} : \boldsymbol{\tau_2} \vdash^{\div} \mathbf{e_2} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{e}_2}{\Delta;\Gamma \vdash^{\div} \mathbf{case}\, \mathbf{v}\, \mathbf{of}\, \mathbf{x_1}.\, \mathbf{e_1} \mid \mathbf{x_2}.\, \mathbf{e_2} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{let}\, \mathsf{x} = \underline{\mathsf{TOSUM}}(\mathsf{v}_u)\, \mathsf{in}\, \mathsf{case}\, \mathsf{x}\, \mathsf{of}\, \mathsf{x}_1.\, \mathsf{e}_1 \mid \mathsf{x}_2.\, \mathsf{e}_2}$$

$$\frac{\Delta;\Gamma \vdash^{+} \mathbf{v} : \langle \boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_n} \rangle \twoheadrightarrow \mathsf{v}_u}{\Delta;\Gamma \vdash^{\div} \mathbf{v.i} : \langle \boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_n} \rangle \twoheadrightarrow \mathsf{let}\, \mathsf{x} = \underline{\mathsf{PRJ}}(\mathsf{i}, \mathsf{v}_u)\, \mathsf{in}\, \underline{\mathsf{RETURN}}(\mathsf{x})}$$

$$\frac{\Delta;\Gamma \vdash^{+} \mathbf{v_1} : \forall[\boldsymbol{\alpha}].\, \boldsymbol{\tau} \to \boldsymbol{\theta} \twoheadrightarrow \mathsf{v}_1 \qquad \Delta;\Gamma \vdash^{+} \mathbf{v_2} : \boldsymbol{\tau}[\boldsymbol{\tau}'/\boldsymbol{\alpha}] \twoheadrightarrow \mathsf{v}_2}{\Delta;\Gamma \vdash^{\div} \mathbf{v_1}\, [\boldsymbol{\tau}']\, \mathbf{v_2} : \boldsymbol{\theta}[\boldsymbol{\tau}'/\boldsymbol{\alpha}] \twoheadrightarrow \mathsf{let}\, \mathsf{x} = \underline{\mathsf{TOFUN}}(\mathsf{v}_1)\, \mathsf{in}\, \mathsf{x}\ \mathsf{v}_2}$$

$$\frac{\Delta;\Gamma \vdash^{+} \mathbf{v} : \mathbf{E}\, \boldsymbol{\tau}_{\mathbf{exn}}\, \boldsymbol{\mu\alpha.\tau} \twoheadrightarrow \mathsf{v}_u}{\Delta;\Gamma \vdash^{\div} \mathbf{unfold}\, \mathbf{v} : \boldsymbol{\tau}[\boldsymbol{\mu\alpha.\tau}/\boldsymbol{\alpha}] \twoheadrightarrow \mathsf{let}\, \mathsf{x} = \underline{\mathsf{TOFOLD}}(\mathsf{v}_u)\, \mathsf{in}\, \underline{\mathsf{RETURN}}(\mathsf{x})}$$

$$\frac{\Delta;\Gamma \vdash^{+} \mathbf{v} : \exists\boldsymbol{\alpha}.\, \boldsymbol{\tau} \twoheadrightarrow \mathsf{v}_u \qquad \Delta, \boldsymbol{\alpha};\Gamma, \mathbf{x} : \boldsymbol{\tau} \vdash^{\div} \mathbf{e} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{e}_u}{\Delta;\Gamma \vdash^{\div} \mathbf{unpack}\, (\boldsymbol{\alpha}, \mathbf{x}) = \mathbf{v}\, \mathbf{in}\, \mathbf{e} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{let}\, \mathsf{x} = \mathsf{v}_u\, \mathsf{in}\, \mathsf{e}_u}$$

$$\frac{\Delta;\Gamma \vdash^{\div} \mathbf{e} : \mathbf{E}\, \boldsymbol{\tau}_{\mathbf{exn}}\, \boldsymbol{\tau} \twoheadrightarrow \mathsf{e} \qquad \Delta;\Gamma, \mathbf{x_1} : \boldsymbol{\tau} \vdash^{\div} \mathbf{e_1} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{e}_1 \qquad \Delta;\Gamma, \mathbf{x_2} : \boldsymbol{\tau}_{\mathbf{exn}} \vdash^{\div} \mathbf{e_2} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{e}_2}{\Delta;\Gamma \vdash^{\div} \mathbf{handle}\, \mathbf{e}\, \mathbf{with}\, (\mathbf{x_1}.\, \mathbf{e_1})\, (\mathbf{x_2}.\, \mathbf{e_2}) : \boldsymbol{\theta} \twoheadrightarrow \begin{array}{l} \mathsf{let}\, \mathsf{x}_r = \mathsf{e}\, \mathsf{in}\, \mathsf{case}\, \mathsf{x}_r\, \mathsf{of} \\ \qquad \mathsf{x}_1 .\, \mathsf{e}_1 \\ \qquad \mathsf{x}_2 .\, \mathsf{e}_2 \end{array}}$$

Figure 28: Relating λ^{ST} terms to λ^{S} terms

9 Back Translation Correctness

$$
\begin{array}{rcl}
\mathrm{Atom}^V[\boldsymbol{\tau}] & \stackrel{\mathrm{def}}{=} & \{(k, \mathsf{v}, \mathbf{v}) \mid k \in \mathbb{N} \wedge \cdot;\cdot \vdash \mathsf{v} : \mathsf{U} \wedge \cdot;\cdot \vdash \mathbf{v} : \boldsymbol{\tau}\} \\
\mathrm{Atom}^R[\boldsymbol{\theta}] & \stackrel{\mathrm{def}}{=} & \{(k, \mathsf{v}, \mathbf{r}) \mid k \in \mathbb{N} \wedge \cdot;\cdot \vdash \mathsf{v} : \mathsf{R} \wedge \cdot;\cdot \vdash \mathbf{r} : \boldsymbol{\theta}\} \\
\mathrm{Atom}^E[\boldsymbol{\theta}] & \stackrel{\mathrm{def}}{=} & \{(k, \mathsf{e}, \mathbf{e}) \mid k \in \mathbb{N} \wedge \cdot;\cdot \vdash \mathsf{e} : \mathsf{R} \wedge \cdot;\cdot \vdash \mathbf{e} : \boldsymbol{\theta}\} \\
\mathrm{Atom}^K[\boldsymbol{\theta}] & \stackrel{\mathrm{def}}{=} & \{(k, K_1, K_2) \mid k \in \mathbb{N} \wedge \exists\theta.\ \vdash K_1 : (\cdot;\cdot \vdash \mathsf{R}) \Rightarrow (\cdot;\cdot \vdash \theta) \wedge \vdash K_2 : (\cdot;\cdot \vdash \boldsymbol{\theta}) \Rightarrow (\cdot;\cdot \vdash \theta)\} \\
\mathrm{Rel}^{\mathsf{U}}[\boldsymbol{\tau}] & \stackrel{\mathrm{def}}{=} & \{R \in \mathscr{P}(\mathrm{Atom}^V[\boldsymbol{\tau}]) \mid \forall j \leq k, \mathsf{v}, \mathbf{v}.\ (k, \mathsf{v}, \mathbf{v}) \in R \implies (j, \mathsf{v}, \mathbf{v}) \in R\}
\end{array}
$$

Figure 29: Universal Type Logical Relation Auxiliary Definitions

$$
\begin{array}{rcl}
\mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}} & \subset & \mathrm{Atom}^V[\rho^{\mathsf{U}}(\boldsymbol{\tau})] \\
\mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\alpha}]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \rho^{\mathsf{U}}_R(\boldsymbol{\alpha}) \\
\mathcal{V}^{\mathsf{U}}[\![\langle\rangle]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \{(k, \underline{\mathsf{UNIT}}, \langle\rangle)\} \\
\mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau_1} + \boldsymbol{\tau_2}]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \{(k, \underline{\mathsf{IN}}(\mathsf{i}, \mathsf{v}_u), \mathbf{inj_i\ v}) \mid \\
& & \quad (k, \mathsf{v}_u, \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau_i}]\!]\rho^{\mathsf{U}}\} \\
\mathcal{V}^{\mathsf{U}}[\![\langle\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, \ldots, \boldsymbol{\tau_n}\rangle]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \{(k, \underline{\mathsf{CONS}}(\mathsf{v}_u, \mathsf{v}'_u), \langle\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\rangle) \mid \\
& & \quad (k, \mathsf{v}_u, \mathbf{v_1}) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau_1}]\!]\rho^{\mathsf{U}} \ \wedge\ (k, \mathsf{v}'_u, \langle\mathbf{v_2}, \ldots, \mathbf{v_n}\rangle) \in \mathcal{V}^{\mathsf{U}}[\![\langle\boldsymbol{\tau_2}, \ldots, \boldsymbol{\tau_n}\rangle]\!]\rho^{\mathsf{U}}\} \\
\mathcal{V}^{\mathsf{U}}[\![\forall[\alpha].\,\boldsymbol{\tau} \to \boldsymbol{\theta}]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \{(k, \underline{\mathsf{LAMBDA}}(\lambda(\mathsf{x}_u : \mathsf{U}).\, \mathsf{e}_u), \boldsymbol{\lambda[\alpha](x : \tau).\, e}) \mid \\
& & \quad \forall \boldsymbol{\tau}', R \in \mathrm{Rel}^{\mathsf{U}}[\rho^{\mathsf{U}}(\boldsymbol{\tau}')], j \leq k, (j, \mathsf{v}_u, \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}\prime}. \\
& & \qquad (j, \mathsf{e}_u[\mathsf{v}_u/\mathsf{x}_u], \mathbf{e}[\boldsymbol{\tau}'/\boldsymbol{\alpha}][\mathbf{v}/\mathbf{x}]) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}\prime} \\
& & \qquad \textit{where. } \rho^{\mathsf{U}\prime} = \rho^{\mathsf{U}}[\boldsymbol{\alpha} \mapsto \boldsymbol{\tau}', R]\} \\
\mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\mu\alpha.\,\tau}]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \{(0, \mathsf{v}_u, \mathbf{v})\} \\
& & \cup \\
& & \{(k+1, \underline{\mathsf{FOLD}}(\mathsf{v}_u), \mathbf{fold}_{\rho^{\mathsf{U}}(\boldsymbol{\mu\alpha.\tau})}\ \mathbf{v}) \mid (k, \mathsf{v}_u, \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}[\boldsymbol{\mu\alpha.\,\tau}/\boldsymbol{\alpha}]]\!]\rho^{\mathsf{U}}\} \\
\mathcal{V}^{\mathsf{U}}[\![\mathbf{0}]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \emptyset \\
\mathcal{V}^{\mathsf{U}}[\![\exists\boldsymbol{\alpha}.\,\boldsymbol{\tau}]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \{(k, \mathsf{v}_u, \mathbf{pack}\ (\boldsymbol{\tau}', \mathbf{v})\ \mathbf{as}\ \rho^{\mathsf{U}}(\exists\boldsymbol{\alpha}.\,\boldsymbol{\tau})) \mid \\
& & \quad \exists R \in \mathrm{Rel}^{\mathsf{U}}[\rho^{\mathsf{U}}(\boldsymbol{\tau}')].\ (k, \mathsf{v}_u, \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}}[\boldsymbol{\alpha} \mapsto \boldsymbol{\tau}', R]\} \\
\mathcal{R}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}} & \subset & \mathrm{Atom}^R[\rho^{\mathsf{U}}(\boldsymbol{\theta})] \\
\mathcal{R}^{\mathsf{U}}[\![\mathbf{E}\,\boldsymbol{\tau_{exn}}\,\boldsymbol{\tau}]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \{(k, \underline{\mathsf{RETURN}}(\mathsf{v}_u), \mathbf{return\ v}) \mid (k, \mathsf{v}_u, \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}}\} \\
& & \cup \\
& & \{(k, \underline{\mathsf{RAISE}}(\mathsf{v}_u), \mathbf{raise\ v}) \mid (k, \mathsf{v}_u, \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau_{exn}}]\!]\rho^{\mathsf{U}}\} \\
\mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}} & \subset & \mathrm{Atom}^E[\rho^{\mathsf{U}}(\boldsymbol{\theta})] \\
\mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \{\,(k, \mathsf{e}_u, \mathbf{e}) \mid \\
& & \quad \forall j \leq k, K_1, K_2.\ (j, K_1, K_2) \in \mathcal{K}\,[\![\boldsymbol{\theta}]\!]\,\rho \implies (j, K_1[\mathsf{e}_u], K_2[\mathbf{e}]) \in \mathcal{O}\} \\
\mathcal{K}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}} & \subset & \mathrm{Atom}^K[\rho^{\mathsf{U}}(\boldsymbol{\theta})] \\
\mathcal{K}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \{\,(k, K_1, K_2) \mid \\
& & \quad \forall j \leq k, \mathsf{v}_u, \mathbf{r}.\ (j, \mathsf{v}_u, \mathbf{r}) \in \mathcal{R}[\![\boldsymbol{\theta}]\!]\rho \implies (j, K_1[\mathsf{v}_u], K_2[\mathbf{r}]) \in \mathcal{O}\} \\
\mathcal{D}^{\mathsf{U}}[\![\cdot]\!] & \stackrel{\mathrm{def}}{=} & \{\,\emptyset\,\} \\
\mathcal{D}^{\mathsf{U}}[\![\Delta, \boldsymbol{\alpha}]\!] & \stackrel{\mathrm{def}}{=} & \{\,\rho^{\mathsf{U}}[\boldsymbol{\alpha} \mapsto \boldsymbol{\tau}, R] \mid\ \rho^{\mathsf{U}} \in \mathcal{D}\,[\![\Delta]\!]\ \wedge\ R \in \mathrm{Rel}^{\mathsf{U}}[\boldsymbol{\tau}]\,\} \\
\mathcal{G}^{\mathsf{U}}[\![\cdot]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \{\,(k, \emptyset) \mid\ k \in \mathbb{N}\,\} \\
\mathcal{G}^{\mathsf{U}}[\![\Gamma, \mathsf{x} : \sigma]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \{\,(k, \gamma^{\mathsf{U}}[\mathsf{x} \mapsto \mathsf{v}_1, \mathsf{v}_2]) \mid\ (k, \gamma^{\mathsf{U}}) \in \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\rho^{\mathsf{U}}\ \wedge\ (k, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}\,[\![\sigma]\!]\,\emptyset\,\} \\
\mathcal{G}^{\mathsf{U}}[\![\Gamma, \mathbf{x} : \boldsymbol{\tau}]\!]\rho^{\mathsf{U}} & \stackrel{\mathrm{def}}{=} & \{(k, \gamma^{\mathsf{U}}[\mathbf{x} \mapsto \mathsf{v}_u, \mathbf{v}]) \mid\ (k, \gamma^{\mathsf{U}}) \in \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\rho^{\mathsf{U}}\ \wedge\ (k, \mathsf{v}_u, \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}}\}
\end{array}
$$

Figure 30: Universal Type Logical Relation

$$
\begin{array}{rcl}
\Delta;\Gamma \vdash \mathsf{v}' \approx^{log}_{\mathcal{V}^{\mathsf{U}}} \mathsf{v} : \sigma & \stackrel{\text{def}}{=} & \mathsf{v}' \in \lambda^{\mathrm{S}} \wedge \forall \rho^{\mathsf{U}} \in \mathcal{D}^{\mathsf{U}}[\![\Delta]\!], (k, \gamma^{\mathsf{U}}) \in \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\rho^{\mathsf{U}}. \\
& & \quad (k, \gamma^{\mathsf{U}}(\mathsf{v}'), \rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathsf{v}))) \in \mathcal{V}\,[\![\sigma]\!]\,\emptyset \\
\Delta;\Gamma \vdash \mathsf{e}' \approx^{log}_{\mathcal{E}^{\mathsf{U}}} \mathsf{e} : \sigma & \stackrel{\text{def}}{=} & \mathsf{e}' \in \lambda^{\mathrm{S}} \wedge \forall \rho^{\mathsf{U}} \in \mathcal{D}^{\mathsf{U}}[\![\Delta]\!], (k, \gamma^{\mathsf{U}}) \in \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\rho^{\mathsf{U}}. \\
& & \quad (k, \gamma^{\mathsf{U}}(\mathsf{e}'), \rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathsf{e}))) \in \mathcal{E}\,[\![\sigma]\!]\,\emptyset \\
\Delta;\Gamma \vdash \mathsf{v}_u \approx^{log}_{\mathcal{V}^{\mathsf{U}}} \mathbf{v} : \boldsymbol{\tau} & \stackrel{\text{def}}{=} & \mathsf{v}_u \in \lambda^{\mathrm{S}} \wedge \forall \rho^{\mathsf{U}} \in \mathcal{D}^{\mathsf{U}}[\![\Delta]\!], (k, \gamma^{\mathsf{U}}) \in \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\rho^{\mathsf{U}}. \\
& & \quad (k, \gamma^{\mathsf{U}}(\mathsf{v}_u), \rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{v}))) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}} \\
\Delta;\Gamma \vdash \mathsf{v}_u \approx^{log}_{\mathcal{R}^{\mathsf{U}}} \mathbf{r} : \boldsymbol{\theta} & \stackrel{\text{def}}{=} & \mathsf{v}_u \in \lambda^{\mathrm{S}} \wedge \forall \rho^{\mathsf{U}} \in \mathcal{D}^{\mathsf{U}}[\![\Delta]\!], (k, \gamma^{\mathsf{U}}) \in \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\rho^{\mathsf{U}}. \\
& & \quad (k, \gamma^{\mathsf{U}}(\mathsf{v}_u), \rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{r}))) \in \mathcal{R}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}} \\
\Delta;\Gamma \vdash \mathsf{e}_u \approx^{log}_{\mathcal{E}^{\mathsf{U}}} \mathbf{e} : \boldsymbol{\theta} & \stackrel{\text{def}}{=} & \mathsf{e}_u \in \lambda^{\mathrm{S}} \wedge \forall \rho^{\mathsf{U}} \in \mathcal{D}^{\mathsf{U}}[\![\Delta]\!], (k, \gamma^{\mathsf{U}}) \in \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\rho^{\mathsf{U}}. \\
& & \quad (k, \gamma^{\mathsf{U}}(\mathsf{e}_u), \rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{e}))) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}
\end{array}
$$

Figure 31: Universal Type Logical Relation for Open Terms

Lemma 9.1 (Universal Type Logical Relation Weakening)
If $\rho^{\mathsf{U}} \in \mathcal{D}[\![\Delta]\!]$, $\Delta \vdash \boldsymbol{\tau}$ *and* $\Delta \vdash \boldsymbol{\theta}$, $\Delta \vdash \boldsymbol{\tau}'$, $\Delta \vdash \Gamma$ *and* $R \in \mathrm{Rel}^{\mathsf{U}}[\boldsymbol{\tau}']$, *then*

1. $\mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}} = \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}\prime}$
2. $\mathcal{R}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}} = \mathcal{R}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}\prime}$
3. $\mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}} = \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}\prime}$
4. $\mathcal{K}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}} = \mathcal{K}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}\prime}$
5. $\mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\rho^{\mathsf{U}} = \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\rho^{\mathsf{U}\prime}$

where $\rho^{\mathsf{U}\prime} = \rho^{\mathsf{U}}[\boldsymbol{\alpha} \mapsto \boldsymbol{\tau}', R]$.

Proof

The first 4 are proven by mutual induction on types. Then the $\mathcal{G}^{\mathsf{U}}[\![\ ]\!]$ case follows by induction on Γ. $\square$

Lemma 9.2 (Universal Type Logical Relation Compositionality)
If $\rho^{\mathsf{U}} \in \mathcal{D}[\![\Delta]\!]$, $\Delta \vdash \boldsymbol{\tau}'$, *and* $R \in \mathrm{Rel}^{\mathsf{U}}[\boldsymbol{\tau}']$, *then if* $\Delta, \boldsymbol{\alpha} \vdash \boldsymbol{\tau}$ *and* $\Delta, \boldsymbol{\alpha} \vdash \boldsymbol{\theta}$,

1. $\mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}\prime} = \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}[\boldsymbol{\alpha}/\boldsymbol{\tau}']]\!]\rho^{\mathsf{U}}$
2. $\mathcal{R}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}\prime} = \mathcal{R}^{\mathsf{U}}[\![\boldsymbol{\tau}[\boldsymbol{\alpha}/\boldsymbol{\tau}']]\!]\rho^{\mathsf{U}}$
3. $\mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}\prime} = \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\tau}[\boldsymbol{\alpha}/\boldsymbol{\tau}']]\!]\rho^{\mathsf{U}}$
4. $\mathcal{K}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}\prime} = \mathcal{K}^{\mathsf{U}}[\![\boldsymbol{\tau}[\boldsymbol{\alpha}/\boldsymbol{\tau}']]\!]\rho^{\mathsf{U}}$

where $\rho^{\mathsf{U}\prime} = \rho^{\mathsf{U}}[\boldsymbol{\alpha} \mapsto \boldsymbol{\tau}', R]$.

Proof

By induction $k, \boldsymbol{\tau}$ and $\boldsymbol{\theta}$, using Lemma 9.1 where appropriate. $\square$

Lemma 9.3 (Monotonicity)
If $j < k$ *then*

1. *If* $(k, \mathsf{v}_u, \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}}$, *then* $(j, \mathsf{v}_u, \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}}$.
2. *If* $(k, \mathsf{v}_u, \mathbf{r}) \in \mathcal{R}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$, *then* $(j, \mathsf{v}_u, \mathbf{r}) \in \mathcal{R}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$.
3. *If* $(k, \mathsf{e}_u, \mathbf{e}) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$, *then* $(j, \mathsf{e}_u, \mathbf{e}) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$.
4. *If* $(k, K_1, K_2) \in \mathcal{K}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$, *then* $(j, K_1, K_2) \in \mathcal{K}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$.
5. *If* $(k, \gamma^{\mathsf{U}}) \in \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\rho^{\mathsf{U}}$, *then* $(j, \gamma^{\mathsf{U}}) \in \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\rho^{\mathsf{U}}$.

Lemma 9.4 (Universal Type Value Relation is Admissible)
$\mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}} \in \mathrm{Rel}^{\mathsf{U}}[\rho^{\mathsf{U}}(\boldsymbol{\tau})]$

Proof

Immediate corollary of Lemma 9.3. $\square$

Lemma 9.5 (Universal Type Logical Relation Monadic Bind)
There are a few different versions, depending on how the two logical relations are interacting, however the proofs are essentially the same.

1. *If* $(k, \mathsf{e}_u, \mathbf{e}) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$ *and for any* $j \leq k, (j, \mathsf{v}, \mathbf{r}) \in \mathcal{R}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$ *and* $(j, K_1[\mathsf{v}], K_2[\mathbf{r}]) \in \mathcal{E}[\![\theta]\!]\rho$ *then* $(k, K_1[\mathsf{e}_u], K_2[\mathbf{e}]) \in \mathcal{E}[\![\theta]\!]\rho$.
2. *If* $(k, e_1, e_2) \in \mathcal{E}[\![\theta]\!]\rho$ *and for any* $j \leq k, (j, r_1, r_2) \in \mathcal{R}[\![\theta]\!]\rho$, $(j, \mathsf{K}[r_1], \mathbf{K}[r_2]) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$, *then* $(\mathsf{K}[e_1], \mathbf{K}[e_2]) \in \mathcal{E}^{\mathsf{U}}[\![\theta]\!]\rho^{\mathsf{U}}$.
3. *If* $(k, \mathsf{e}_u, \mathbf{e}) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$ *and for any* $j \leq k, (j, \mathsf{v}, \mathbf{r}) \in \mathcal{R}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$ *and* $(j, \mathsf{K}[\mathsf{v}], \mathbf{K}[\mathbf{r}]) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}']\!]\rho^{\mathsf{U}}$ *then* $(k, \mathsf{K}[\mathsf{e}_u], \mathbf{K}[\mathbf{e}]) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}']\!]\rho^{\mathsf{U}}$.

Proof

We present a proof of the first case, the others are essentially the same. Let $(k, K_1', K_2') \in \mathcal{K}[\![\theta]\!]\rho$. We want to show that $(k, K_1'[K_1[\mathsf{e}_u]], K_2'[K_2[\mathbf{e}]]) \in \mathcal{O}$. By definition of $\mathcal{E}^{\mathsf{U}}[\![\ ]\!]$, it is sufficient to show that $(k, K_1'[K_1], K_2'[K_2]) \in \mathcal{K}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$.

So, let $j \leq k, (j, \mathsf{v}, \mathbf{r}) \in \mathcal{R}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$, we need to show that $(k, K_1'[K_1[\mathsf{v}]], K_2'[K_2[\mathbf{r}]]) \in \mathcal{O}$. By Lemma 7.6, $(j, K_1', K_2') \in \mathcal{K}[\![\theta]\!]\rho$, so the result follows from the assumption and that $(j, K_1[\mathsf{v}], K_2[\mathbf{r}]) \in \mathcal{E}[\![\theta]\!]\rho$. ☐

Lemma 9.6 (Universal Type Logical Relation Anti-reduction)
If $\mathsf{e}_u \longmapsto^{k_u} \mathsf{e}_u'$ *and* $\mathbf{e} \longmapsto^{k_t} \mathbf{e}'$ *and* $k \leq \min(k_u, k_t) + k'$ *then if* $(k', \mathsf{e}_u', \mathbf{e}') \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$, *then* $(k, \mathsf{e}_u, \mathbf{e}) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$.

Proof

Direct from definition of $\mathcal{O}$. ☐

Lemma 9.7 (Universal Type Derived Computation Rules)
For appropriately typed expressions,

1. $\underline{\mathsf{TOSUM}}(\underline{\mathsf{IN}}(\mathsf{i}, \mathsf{v}_u)) \longmapsto^* \mathsf{inj_i}\, \mathsf{v}_u$
2. $\underline{\mathsf{TOPAIR}}(\underline{\mathsf{CONS}}(\mathsf{v}_u, \mathsf{v}_u')) \longmapsto^* \langle \mathsf{v}_u, \mathsf{v}_u' \rangle$
3. $\underline{\mathsf{TOFUN}}(\underline{\mathsf{LAMBDA}}(\lambda(\mathsf{x}_u : \mathsf{U}).\, \mathsf{e}_u)) \longmapsto^* \lambda(\mathsf{x}_u : \mathsf{U}).\, \mathsf{e}_u$
4. $\underline{\mathsf{TOFOLD}}(\underline{\mathsf{FOLD}}(\mathsf{v}_u)) \longmapsto^{\geq 1} \mathsf{v}_u$

Proof

Trivial. ☐

Lemma 9.8 (Correctness of Fix)
If $\cdot;\cdot \vdash \mathsf{v}_f : (\sigma_1 \to \sigma_2) \to \sigma_1 \to \sigma_2$ *and* $\cdot;\cdot \vdash \mathsf{v}_{arg} : \sigma_1$, *then*

$$\underline{\mathsf{FIX}}_{\sigma_1 \to \sigma_2}(\mathsf{v}_f)\ \mathsf{v}_{arg} \longmapsto^* \mathsf{let}\, \mathsf{x}_f = \mathsf{v}_f\ \underline{\mathsf{FIX}}_{\sigma_1 \to \sigma_2}(\mathsf{v}_f)\, \mathsf{in}\, \mathsf{x}_f\ \mathsf{v}_{arg}$$

Proof

Straightforward calculation. ☐

Lemma 9.9 (Embed/Project Unroll)

$$\underline{\mathsf{EP}}(\emptyset, \mu\alpha.\, \sigma)\ \langle\rangle \longmapsto^* \langle \begin{array}{l} \lambda(\mathsf{x} : \delta_\sigma(\mu\alpha.\, \sigma)). \\ \mathsf{let}\, \mathsf{y} = \mathsf{unfold}\, \mathsf{x}\, \mathsf{in} \\ \mathsf{let}\, \mathsf{y}_u = \underline{\mathsf{EMBED}}(\cdot, \sigma[\alpha/\mu\alpha.\, \sigma])\ \mathsf{y}\, \mathsf{in} \\ \underline{\mathsf{FOLD}}(\mathsf{y}_u) \end{array}, \begin{array}{l} \lambda(\mathsf{x}_u : \mathsf{U}). \\ \mathsf{let}\, \mathsf{y}_u = \underline{\mathsf{TOFOLD}}(\mathsf{x}_u)\, \mathsf{in} \\ \mathsf{let}\, \mathsf{y} = \underline{\mathsf{PROJECT}}(\cdot, \sigma[\alpha/\mu\alpha.\, \sigma])\ \mathsf{y}_u\, \mathsf{in} \\ \mathsf{fold}_{\mu\alpha.\sigma}\, \mathsf{y} \end{array} \rangle.$$

Proof

The result is a simple consequence of Lemma 9.8 and the following lemma:

1. $\underline{\mathsf{EMBED}}(\delta[\alpha \mapsto \mu\alpha.\, \sigma, \mathsf{x}_{\mu\alpha.\sigma}], \sigma')[\underline{\mathsf{EP}}(\delta, \mu\alpha.\, \sigma)/\mathsf{x}_{\mu\alpha.\sigma}] = \underline{\mathsf{EMBED}}(\delta, \sigma'[\mu\alpha.\, \sigma/\alpha])$

2. $\underline{\mathsf{PROJECT}}(\delta[\alpha \mapsto \mu\alpha.\,\sigma, x_{\mu\alpha.\sigma}], \sigma')[\underline{\mathsf{EP}}(\delta, \mu\alpha.\,\sigma)/x_{\mu\alpha.\sigma}] = \underline{\mathsf{PROJECT}}(\delta, \sigma'[\mu\alpha.\,\sigma/\alpha])$

which holds by a straightforward induction on σ'. $\square$

Theorem 9.10 (Interpret = Interoperate)

1. *If* $(k, e_u, \mathbf{e}) \in \mathcal{E}^{\mathsf{U}}[\![\sigma^{\div}]\!]\emptyset$*, then* $(k, \mathsf{let}\, x = e_u\, \mathsf{in}\, \underline{\mathsf{PROJECT}}(\sigma)\ x, {}^{\sigma}\mathcal{ST}\,\mathbf{e}) \in \mathcal{E}\,[\![\sigma]\!]\,\emptyset$.
2. *If* $(k, e, e') \in \mathcal{E}\,[\![\sigma]\!]\,\emptyset$*, then* $(k, \mathsf{let}\, x = e\, \mathsf{in}\, \underline{\mathsf{EMBED}}(\sigma)\ x, \mathcal{TS}^{\,\sigma}\, e') \in \mathcal{E}^{\mathsf{U}}[\![\sigma^{\div}]\!]\emptyset$.
3. *If* $(k, v, \mathbf{r}) \in \mathcal{R}^{\mathsf{U}}[\![\sigma^{\div}]\!]\emptyset$*, then then either*
   $\underline{\mathsf{PROJECT}}(\sigma)\ v \longmapsto^{k}$ *and* ${}^{\sigma}\mathcal{ST}\,\mathbf{r} \longmapsto^{k}$*, or*
   $\underline{\mathsf{PROJECT}}(\sigma)\ v \longmapsto^{*} v'_1$, ${}^{\sigma}\mathcal{ST}\,\mathbf{r} \longmapsto^{*} v'_2$ *and* $(k, v'_1, v'_2) \in \mathcal{R}[\![\sigma]\!]\emptyset$.
4. *If* $(k, v_u, \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![\sigma^{+}]\!]\emptyset$*, then either*
   $\underline{\mathsf{PROJECT}}(\cdot, \sigma)\ v_u \longmapsto^{k}$ *and* ${}^{\sigma}\mathcal{ST}\,\mathbf{return\ v} \longmapsto^{k}$*, or*
   $\underline{\mathsf{PROJECT}}(\cdot, \sigma)\ v_u \longmapsto^{*} v$, ${}^{\sigma}\mathcal{ST}\,\mathbf{return\ v} \longmapsto^{*} v'$ *and* $(k, v, v') \in \mathcal{V}\,[\![\sigma]\!]\,\emptyset$.
5. *If* $(k, v, v') \in \mathcal{V}\,[\![\sigma]\!]\,\emptyset$*, then either*
   $\underline{\mathsf{EMBED}}(\cdot, \sigma) v \longmapsto^{k}$ *and* $\mathcal{TS}^{\,\sigma}\, v' \longmapsto^{k}$ *or* $\underline{\mathsf{EMBED}}(\cdot, \sigma)\ v \longmapsto^{*} \underline{\mathsf{RETURN}}(v_u)$, $\mathcal{TS}^{\,\sigma}\, v' \longmapsto^{*} \mathbf{return\ v}$
   and $(k, v_u, \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![\sigma^{+}]\!]\emptyset$.

Proof

The first 2 cases follow from the latter cases. The third case follows from the later ones and the interpretation of $\mathbf{0}$.

For the last 2 cases, we proceed by nested induction on k, σ.

Case $(k, v_u, \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![\sigma^{+}]\!]\emptyset$:

Case 1: trivial.

Case $\sigma_1 + \sigma_2$: $v_u = \underline{\mathsf{IN}}(\mathsf{i}, v'_u)$ and $\mathbf{v} = \mathbf{inj_i\ v'}$. By Lemma 9.7, $\underline{\mathsf{PROJECT}}(\cdot, \sigma_1 + \sigma_2)\ \underline{\mathsf{IN}}(\mathsf{i}, v'_u) \longmapsto^{*} \mathsf{let}\, x {=} \underline{\mathsf{PROJECT}}(\cdot, \sigma_i)\ v'_u\ \mathsf{in}\ \mathsf{inj_i}\, x$. Next, ${}^{\sigma_1+\sigma_2}\mathcal{ST}\,\mathbf{inj_i\ v'} \longmapsto^{*} \mathsf{let}\, x = {}^{\sigma_i}\mathcal{ST}\,\mathbf{return\ v'}\ \mathsf{in}\ \mathsf{inj_i}\, x$, so the result follows by inductive hypothesis and Lemma 9.6.

Case $\sigma_1 \times \sigma_2$: By Lemma 9.7 and inductive hypothesis.

Case $\sigma_1 {\to} \sigma_2$: $v_u = \underline{\mathsf{CONS}}(\underline{\mathsf{LAMBDA}}(\lambda(x_u : \mathsf{U}).\, e_u), \underline{\mathsf{CONS}}(v_{env}, \underline{\mathsf{UNIT}}))$ and $\mathbf{v} = \mathbf{pack}\,(\boldsymbol{\tau}, \langle \boldsymbol{\lambda}(\mathbf{x} : \langle \boldsymbol{\tau}', \sigma_1{}^{+}\rangle).\,\mathbf{e}, \mathbf{v_{env}}\rangle$ and there exists $R \in \mathrm{Atom}^{V}[\boldsymbol{\tau}]$ such that $(k, v_{env}, \mathbf{v_{env}}) \in R$ and $(k, \underline{\mathsf{LAMBDA}}(\lambda(x_u : \mathsf{U}).\, e_u), \boldsymbol{\lambda}(\mathbf{x} : \langle \boldsymbol{\tau}', \sigma_1{}^{+}\rangle).\,\mathbf{e})$
$\mathcal{V}^{\mathsf{U}}[\![\langle \boldsymbol{\alpha}, \sigma_1{}^{+}\rangle \to \sigma_2{}^{\div}]\!]\rho^{\mathsf{U}\prime}$ where $\rho^{\mathsf{U}\prime} = \rho^{\mathsf{U}}[\emptyset \mapsto \boldsymbol{\alpha}, \boldsymbol{\tau}]R$.
First,

$$\begin{aligned}\underline{\mathsf{PROJECT}}(\cdot, \sigma_1 \to \sigma_2)\ v_u \longmapsto^{*} \lambda(y : \delta_{\sigma}(\sigma_1)).\ & \mathsf{let}\, y_u = \underline{\mathsf{EMBED}}(\cdot, \sigma_1)\ y\ \mathsf{in} \\ & \mathsf{let}\, x = \underline{\mathsf{CONS}}(v_{env}, \underline{\mathsf{CONS}}(y_u, \underline{\mathsf{UNIT}}))\ \mathsf{in} \\ & \mathsf{let}\, x_r = \underline{\mathsf{LAMBDA}}(\lambda(x_u : \mathsf{U}).\, e_u)\ x\ \mathsf{in} \\ & \mathsf{let}\, x''_u = \underline{\mathsf{TOLHS}}(x_r)\ \mathsf{in} \\ & \underline{\mathsf{PROJECT}}(\cdot, \sigma_2)\ x''_u\end{aligned}$$

and

$${}^{\sigma_1\to\sigma_2}\mathcal{ST}\,\mathbf{return\ v} \longmapsto^{*} \lambda(x : \sigma_1).\ {}^{\sigma_2}\mathcal{ST}\left(\begin{aligned}\mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{z}) = \mathbf{v}\ \mathbf{in}\ & \mathbf{let}\, \mathbf{x_f} = \mathbf{z.1}\ \mathbf{in} \\ & \mathbf{let}\, \mathbf{x_{env}} = \mathbf{z.2}\ \mathbf{in} \\ & \mathbf{let}\, \mathbf{x} = \mathcal{TS}^{\,\sigma_1}\, x\ \mathbf{in}\ \mathbf{x_f}\ [\boldsymbol{\alpha}]\ \langle \mathbf{x_{env}}, \mathbf{x}\rangle\end{aligned}\right)$$

.

Let $j \leq k$ and $(j, \mathsf{v}_{larg}, \mathsf{v}_{rarg}) \in \mathcal{V}[\![\sigma_1]\!]\,\emptyset$. By Lemma 7.11, it is sufficient to show that $(j,$

$\mathsf{let}\ \mathsf{y}_u = \underline{\mathsf{EMBED}}(\cdot, \sigma_1)\ \mathsf{v}_{larg}\ \mathsf{in}$,
$\mathsf{let}\ \mathsf{x} = \underline{\mathsf{CONS}}(\mathsf{v}_{env}, \underline{\mathsf{CONS}}(\mathsf{y}_u, \underline{\mathsf{UNIT}}))\ \mathsf{in}$
$\mathsf{let}\ \mathsf{x}_r = \underline{\mathsf{LAMBDA}}(\lambda(\mathsf{x}_u : \mathsf{U}).\, \mathsf{e}_u)\ \mathsf{x}\ \mathsf{in}$
$\mathsf{let}\ \mathsf{x}''_u = \underline{\mathsf{TOLHS}}(\mathsf{x}_r)\ \mathsf{in}$
$\underline{\mathsf{PROJECT}}(\cdot, \sigma_2)\ \mathsf{x}''_u$

${}^{\sigma_2}\mathcal{ST}\ \mathbf{let}\ \mathbf{x} = \mathcal{TS}^{\sigma_1}\ \mathsf{v}_{rarg}\ \mathbf{in}\ (\boldsymbol{\lambda}(\mathbf{x} : \langle \tau', \sigma_1{}^+\rangle).\, \mathbf{e})\ \langle \mathbf{v_{env}}, \mathbf{x}\rangle) \in \mathcal{E}[\![\sigma_2]\!]\,\emptyset$.

By inductive hypothesis either both $\underline{\mathsf{EMBED}}(\cdot, \sigma_1)\ \mathsf{v}_{larg} \longmapsto^j$ and $\mathcal{TS}^{\sigma_1}\ \mathsf{v}_{rarg} \longmapsto^j$ and we're done, or $\underline{\mathsf{EMBED}}(\cdot, \sigma_1)\ \mathsf{v}_{larg} \longmapsto^* \mathsf{v}_{uarg}$ and $\mathcal{TS}^{\sigma_1}\ \mathsf{v}_{rarg} \longmapsto^* \mathbf{return}\ \mathbf{v_{targ}}$ where $(j, \mathsf{v}_{uarg}, \mathbf{v_{targ}}) \in \mathcal{V}^{\mathsf{U}}[\![\sigma_1{}^+]\!]\emptyset$. So it is sufficient to show

$(j, \mathsf{let}\ \mathsf{x}_r = \underline{\mathsf{LAMBDA}}(\lambda(\mathsf{x}_u : \mathsf{U}).\, \mathsf{e}_u)\ \underline{\mathsf{CONS}}(\mathsf{v}_{env}, \underline{\mathsf{CONS}}(\mathsf{v}_{uarg}, \underline{\mathsf{UNIT}}))\ \mathsf{in},$
$\quad \mathsf{let}\ \mathsf{x}''_u = \underline{\mathsf{TOLHS}}(\mathsf{x}_r)\ \mathsf{in}$
$\quad \underline{\mathsf{PROJECT}}(\cdot, \sigma_2)\ \mathsf{x}''_u$

${}^{\sigma_2}\mathcal{ST}\ (\boldsymbol{\lambda}(\mathbf{x} : \langle \tau', \sigma_1{}^+\rangle).\, \mathbf{e})\ \langle \mathbf{v_{env}}, \mathbf{v_{targ}}\rangle) \in \mathcal{E}[\![\sigma_2]\!]\,\emptyset$

Next, $(j, \underline{\mathsf{CONS}}(\mathsf{v}_{env}, \underline{\mathsf{CONS}}(\mathsf{v}_{uarg}, \underline{\mathsf{UNIT}})), \langle \mathbf{v_{env}}, \mathbf{v_{targ}}\rangle) \in \mathcal{V}^{\mathsf{U}}[\![\langle \boldsymbol{\alpha}, \sigma_1{}^+\rangle]\!]\rho^{\mathsf{U}\prime}$ by assumption and Lemma 9.3. Then by Lemma 9.5, it is sufficient to show for any $l \leq j$, $(l, \mathsf{v}_r, \mathbf{r}) \in \mathcal{R}[\![\sigma_2{}^{\div}]\!]\emptyset$,

$$(j, \begin{array}{l}\mathsf{let}\ \mathsf{x}''_u = \underline{\mathsf{TOLHS}}(\mathsf{v}_r)\ \mathsf{in}, \\ \underline{\mathsf{PROJECT}}(\cdot, \sigma_2)\ \mathsf{x}''_u\end{array} {}^{\sigma_2}\mathcal{ST}\ \mathbf{r}) \in \mathcal{E}[\![\sigma]\!]\,\emptyset.$$

Which follows by inductive hypothesis and the definition of $\mathcal{V}^{\mathsf{U}}[\![\mathbf{0}]\!]\emptyset$.

Case $\mu\alpha.\, \sigma$: Either $k = 0$ and we're done or there is some k' such that $k = k' + 1$. In the latter case, we have $\mathsf{v}_u = \underline{\mathsf{FOLD}}(\mathsf{v}'_u)$, $\mathbf{v} = \mathbf{fold}_{\boldsymbol{\mu\alpha}.\sigma^+}\ \mathbf{v}'$ where $(k', \mathsf{v}'_u, \mathbf{v}') \in \mathcal{V}^{\mathsf{U}}[\![\sigma^+[\boldsymbol{\mu\alpha}.\, \sigma^+/\boldsymbol{\alpha}]]\!]\emptyset$. By Lemma 9.9 and further calculation,

$$\underline{\mathsf{PROJECT}}(\cdot, \mu\alpha.\, \sigma)\ \underline{\mathsf{FOLD}}(\mathsf{v}'_u) \longmapsto^{\geq 1} \begin{array}{l}\mathsf{let}\ \mathsf{y} = \underline{\mathsf{PROJECT}}(\cdot, \sigma[\alpha/\mu\alpha.\, \sigma])\ \mathsf{v}'_u\ \mathsf{in} \\ \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{y}\end{array}$$

and

$${}^{\mu\alpha.\sigma}\mathcal{ST}\ \mathbf{return}\ \mathbf{fold}_{\boldsymbol{\mu\alpha}.\sigma^+}\ \mathbf{v}' \longmapsto^* \mathsf{let}\ \mathsf{x} = {}^{\sigma[\mu\alpha.\sigma/\alpha]}\mathcal{ST}\ \mathbf{v}'\ \mathsf{in}\ \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{x}$$

Then by inductive hypothesis either both $\underline{\mathsf{PROJECT}}(\cdot, \sigma[\alpha/\mu\alpha.\, \sigma])\ \mathsf{v}'_u \longmapsto^{k'}$ and ${}^{\sigma[\mu\alpha.\sigma/\alpha]}\mathcal{ST}\ \mathbf{v}' \longmapsto^{k'}$, or $\underline{\mathsf{PROJECT}}(\cdot, \sigma[\alpha/\mu\alpha.\, \sigma])\ \mathsf{v}'_u \longmapsto^* \mathsf{v}_l$ and ${}^{\sigma[\mu\alpha.\sigma/\alpha]}\mathcal{ST}\ \mathbf{v}' \longmapsto^* \mathsf{v}_r$ and $(k', \mathsf{v}_l, \mathsf{v}_r) \in \mathcal{V}[\![\sigma[\mu\alpha.\, \sigma/\alpha]]\!]\,\emptyset$. Then we have

$$\begin{array}{l}\mathsf{let}\ \mathsf{y} = \underline{\mathsf{PROJECT}}(\cdot, \sigma[\alpha/\mu\alpha.\, \sigma])\ \mathsf{v}'_u\ \mathsf{in} \\ \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{y}\end{array} \longmapsto^* \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{v}_l$$

and

$$\mathsf{let}\ \mathsf{x} = {}^{\sigma[\mu\alpha.\sigma/\alpha]}\mathcal{ST}\ \mathbf{v}'\ \mathsf{in}\ \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{x} \longmapsto^* \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{v}_r$$

and we have $(k' + 1, \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{v}_l, \mathsf{fold}_{\mu\alpha.\sigma}\ \mathsf{v}_r) \in \mathcal{V}[\![\mu\alpha.\, \sigma]\!]\,\emptyset$.

Case $(k, \mathsf{v}, \mathsf{v}') \in \mathcal{V}[\![\sigma]\!]\,\emptyset$:

Case 1: trivial.

Case $\sigma_1 + \sigma_2$: Then $\mathsf{v}_1 = \mathsf{inj_i}\ \mathsf{v}_{i,1}$, $\mathsf{v}_2 = \mathsf{inj_i}\ \mathsf{v}_{i,2}$ where $(k, \mathsf{v}_{i,1}, \mathsf{v}_{i,2}) \in \mathcal{V}[\![\sigma_i]\!]\,\emptyset$. Next by Lemma 9.7,

$$\underline{\mathsf{EMBED}}(\cdot, \sigma_1 + \sigma_2)\ (\mathsf{inj_i}\ \mathsf{v}_{i,1}) \longmapsto^* \begin{array}{l}\mathsf{let}\ \mathsf{x}' = \underline{\mathsf{EMBED}}(\cdot, \sigma_i)\ \mathsf{v}_{i,1}\ \mathsf{in} \\ \underline{\mathsf{IN}}(\mathsf{i}, \mathsf{x}')\end{array}$$

and

$$\mathcal{TS}^{\sigma_1+\sigma_2}\ (\mathsf{inj_i}\ \mathsf{v}_{i,2}) \longmapsto^* \mathbf{let}\ \mathbf{x} = \mathcal{TS}^{\sigma_i}\ \mathsf{v}_{i,2}\ \mathbf{in}\ \mathbf{return}\ \mathbf{inj_i}\ \mathbf{x}$$

so the result holds by inductive hypothesis and Lemma 9.6.

Case $\sigma_1 \times \sigma_2$: By straightforward computation and inductive hypothesis.

Case $\sigma \to \sigma'$: Then $\mathsf{v}_1 = \lambda(\mathsf{x}_1 : \sigma).\, \mathsf{e}_1$ and $\mathsf{v}_2 = \lambda(\mathsf{x}_2 : \sigma).\, \mathsf{e}_1$. Next,

$$\underline{\mathsf{EMBED}}(\cdot, \sigma \to \sigma')\ (\lambda(\mathsf{x}_1 : \sigma).\, \mathsf{e}_1) \longmapsto^* \underline{\mathsf{CONS}}(\lambda(\mathsf{x}_u : \mathsf{U}).\, \begin{array}{l} \mathsf{let}\, \mathsf{x}'_u = \underline{\mathsf{PRJ}}(2, \mathsf{x}_u)\ \mathsf{in} \\ \mathsf{let}\, \mathsf{x} = \underline{\mathsf{PROJECT}}(\cdot, \sigma)\ \mathsf{x}'_u\ \mathsf{in} \\ \mathsf{let}\, \mathsf{y} = \lambda(\mathsf{x}_1 : \sigma).\, \mathsf{e}_1\ \mathsf{x}\, \mathsf{in} \\ \mathsf{let}\, \mathsf{x}''_u = \underline{\mathsf{EMBED}}(\cdot, \sigma')\ \mathsf{y}\, \mathsf{in} \\ \underline{\mathsf{RETURN}}(\mathsf{x}''_u) \end{array}, \underline{\mathsf{CONS}}(\underline{\mathsf{UNIT}}, \underline{\mathsf{UNIT}}))$$

and

$$\mathbf{return\ pack}\, (\mathbf{1}, \langle \lambda(\mathbf{z} : \langle \mathbf{1}, \sigma^+ \rangle).\, \mathcal{TS}^{\sigma'} \left(\begin{array}{l} \mathsf{let}\, \mathsf{x} = {}^{\sigma}\mathcal{ST}\ \mathbf{z.2}\ \mathsf{in} \\ \quad \lambda(\mathsf{x}_2 : \sigma).\, \mathsf{e}_2\ \mathsf{x} \end{array} \right), \langle\rangle \rangle)\ \mathbf{as}\, (\sigma \to \sigma')^+$$

For $\boldsymbol{\tau}$, we select $\langle\rangle$ and for R we select $\mathrm{Atom}^V[\langle\rangle]$, which is obviously in $\mathrm{Rel}^{\mathsf{U}}[\langle\rangle]$ and $(k, \underline{\mathsf{UNIT}}, \langle\rangle) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\alpha}]\!](\emptyset[\boldsymbol{\alpha} \mapsto \langle\rangle, \mathrm{Atom}^V[\langle\rangle]]) = \mathrm{Atom}^V[\langle\rangle]$ as needed. Let $j \leq k$ and $(j, \underline{\mathsf{CONS}}(\mathsf{v}'_u, \underline{\mathsf{CONS}}(\mathsf{v}_u, \underline{\mathsf{UNIT}})), \langle \mathbf{v}', \mathbf{v} \rangle) \in \mathcal{V}^{\mathsf{U}}[\![\langle \boldsymbol{\alpha}, \sigma^+ \rangle]\!](\emptyset[\boldsymbol{\alpha} \mapsto \langle\rangle, \mathrm{Atom}^V[\langle\rangle]])$. Then by Lemma 9.7 and Lemma 9.6, it is sufficient to show that

$$(j, \begin{array}{l} \mathsf{let}\, \mathsf{x} = \underline{\mathsf{PROJECT}}(\cdot, \sigma)\ \mathsf{v}_u\ \mathsf{in}, \\ \mathsf{let}\, \mathsf{y} = \lambda(\mathsf{x}_1 : \sigma).\, \mathsf{e}_1\ \mathsf{x}\, \mathsf{in} \\ \mathsf{let}\, \mathsf{x}''_u = \underline{\mathsf{EMBED}}(\cdot, \sigma')\ \mathsf{y}\, \mathsf{in} \\ \underline{\mathsf{RETURN}}(\mathsf{x}''_u) \end{array} \mathcal{TS}^{\sigma'} \left(\begin{array}{l} \mathsf{let}\, \mathsf{x} = {}^{\sigma}\mathcal{ST}\ \mathbf{return\ v}\ \mathsf{in} \\ \quad \lambda(\mathsf{x}_2 : \sigma).\, \mathsf{e}_2\ \mathsf{x} \end{array} \right)) \in \mathcal{E}^{\mathsf{U}}[\![\sigma'^{\div}]\!]\emptyset$$

By inductive hypothesis either both $\underline{\mathsf{PROJECT}}(\cdot, \sigma)\ \mathsf{v}_u \longmapsto^k$ and ${}^{\sigma}\mathcal{ST}\ \mathbf{return\ v} \longmapsto^k$, or $\underline{\mathsf{PROJECT}}(\cdot, \sigma)\ \mathsf{v}_u \longmapsto^* \mathsf{v}_l$ and ${}^{\sigma}\mathcal{ST}\ \mathbf{return\ v} \longmapsto^* \mathsf{v}_r$ and $(j, \mathsf{v}_l, \mathsf{v}_r) \in \mathcal{V}[\![\sigma]\!]\, \emptyset$.
Then $(j, \mathsf{e}_1[\mathsf{x}_1/\mathsf{v}_l], \mathsf{e}_2[\mathsf{x}_2/\mathsf{v}_r]) \in \mathcal{E}[\![\sigma']\!]\, \emptyset$, so by Lemma 9.5, Lemma 9.6 and computation, it is sufficient to show that for any $j' \leq j, (j', \mathsf{v}_{l,2}, \mathsf{v}_{r,2}) \in \mathcal{V}[\![\sigma']\!]\, \emptyset$,

$$(j', \begin{array}{l} \mathsf{let}\, \mathsf{x}''_u = \underline{\mathsf{EMBED}}(\cdot, \sigma')\ \mathsf{v}_{l,2}\ \mathsf{in}, \\ \underline{\mathsf{RETURN}}(\mathsf{x}''_u) \end{array} \mathcal{TS}^{\sigma'}\ \mathsf{v}_{r,2}) \in \mathcal{E}^{\mathsf{U}}[\![\sigma'^{+}]\!]\emptyset$$

which follows by inductive hypothesis.

Case $\mu\alpha.\, \sigma$: If $k = 0$, we're done. Otherwise $k = k' + 1$, $\mathsf{v} = \mathsf{fold}_{\mu\alpha.\sigma}\, \mathsf{v}_l$ and $\mathsf{v}' = \mathsf{fold}_{\mu\alpha.\sigma}\, \mathsf{v}_r$. By Lemma 9.9 and further calculation,

$$\underline{\mathsf{EMBED}}(\cdot, \mu\alpha.\, \sigma)\ \mathsf{fold}_{\mu\alpha.\sigma}\, \mathsf{v}_l \longmapsto^{\geq 1} \begin{array}{l} \mathsf{let}\, \mathsf{y}_u = \underline{\mathsf{EMBED}}(\cdot, \sigma[\mu\alpha.\, \sigma/\alpha])\ \mathsf{v}_l\ \mathsf{in} \\ \underline{\mathsf{FOLD}}(\mathsf{y}_u) \end{array}$$

and

$$\mathcal{TS}^{\mu\alpha.\sigma}\ \mathsf{fold}_{\mu\alpha.\sigma}\, \mathsf{v}_r \longmapsto^{\geq 1} \mathbf{let\ x} = \mathcal{TS}^{\sigma[\mu\alpha.\sigma/\alpha]}\ \mathsf{v}_r\ \mathbf{in\ return\ fold}_{(\mu\alpha.\sigma)^+}\ \mathbf{x}$$

By inductive hypothesis either both $\underline{\mathsf{EMBED}}(\cdot, \sigma[\mu\alpha.\, \sigma/\alpha])\ \mathsf{v}_l \longmapsto^k$ and $\mathcal{TS}^{\sigma[\mu\alpha.\sigma/\alpha]}\ \mathsf{v}_r \longmapsto^k$, or $\underline{\mathsf{EMBED}}(\cdot, \sigma[\mu\alpha.\, \sigma/\alpha])\ \mathsf{v}_l \longmapsto^* \mathsf{v}_u$ and $\mathcal{TS}^{\sigma[\mu\alpha.\sigma/\alpha]}\ \mathsf{v}_r \longmapsto^* \mathbf{return\ v}$ and $(k', \mathsf{v}_u, \mathbf{v}) \in \mathcal{V}[\![\sigma[\mu\alpha.\, \sigma/\alpha]]\!]\, \emptyset$. Thus, we have

$$\begin{array}{l} \mathsf{let}\, \mathsf{y}_u = \underline{\mathsf{EMBED}}(\cdot, \sigma[\mu\alpha.\, \sigma/\alpha])\ \mathsf{v}_l\ \mathsf{in} \\ \underline{\mathsf{FOLD}}(\mathsf{y}_u) \end{array} \longmapsto^* \underline{\mathsf{FOLD}}(\mathsf{v}_u)$$

and

$$\mathbf{let\ x} = \mathcal{TS}^{\sigma[\mu\alpha.\sigma/\alpha]}\ \mathsf{v}_r\ \mathbf{in\ return\ fold}_{(\mu\alpha.\sigma)^+}\ \mathbf{x} \longmapsto^* \mathbf{return\ fold}_{(\mu\alpha.\sigma)^+}\ \mathbf{v}$$

and we have $(k' + 1, \underline{\mathsf{FOLD}}(\mathsf{v}_u), \mathbf{fold}_{(\mu\alpha.\sigma)^+}\ \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![(\mu\alpha.\, \sigma)^+]\!]\emptyset$.

□

Theorem 9.11 (Interpreter Fundamental Property)

1. *If* $\Delta;\Gamma \vdash \mathsf{v} : \sigma$ *and* $\Delta;\Gamma \vdash \mathsf{v} : \sigma \twoheadrightarrow \mathsf{v}'$*, then* $\Delta;\Gamma \vdash \mathsf{v}' \approx^{log}_{\mathcal{V}^{\mathsf{U}}} \mathsf{v} : \sigma$*.*
2. *If* $\Delta;\Gamma \vdash \mathsf{e} : \sigma$ *and* $\Delta;\Gamma \vdash \mathsf{e} : \sigma \twoheadrightarrow \mathsf{e}'$*, then* $\Delta;\Gamma \vdash \mathsf{e}' \approx^{log}_{\mathcal{E}^{\mathsf{U}}} \mathsf{e} : \sigma$
3. *If* $\Delta;\Gamma \vdash \mathbf{v} : \boldsymbol{\tau}$ *and* $\Delta;\Gamma \vdash^{+} \mathbf{v} : \boldsymbol{\tau} \twoheadrightarrow \mathsf{v}_u$*, then* $\Delta;\Gamma \vdash \mathsf{v}_u \approx^{log}_{\mathcal{V}^{\mathsf{U}}} \mathbf{v} : \boldsymbol{\tau}$*.*
4. *If* $\Delta;\Gamma \vdash \mathbf{r} : \boldsymbol{\theta}$ *and* $\Delta;\Gamma \vdash^{\div} \mathbf{r} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{v}_u$*, then* $\Delta;\Gamma \vdash \mathsf{v}_u \approx^{log}_{\mathcal{R}^{\mathsf{U}}} \mathbf{r} : \boldsymbol{\theta}$*.*
5. *If* $\Delta;\Gamma \vdash \mathbf{e} : \boldsymbol{\theta}$ *and* $\Delta;\Gamma \vdash^{\div} \mathbf{e} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{e}_u$*, then* $\Delta;\Gamma \vdash \mathsf{e}_u \approx^{log}_{\mathcal{E}^{\mathsf{U}}} \mathbf{e} : \boldsymbol{\theta}$*.*

Proof

By induction over the implicit k and mutual induction over typing/translation derivations. For each case let $\rho^{\mathsf{U}} \in \mathcal{D}^{\mathsf{U}}[\![\Delta]\!]$ and $(k, \gamma^{\mathsf{U}}) \in \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\rho^{\mathsf{U}}$.

Case $\Delta;\Gamma \vdash \mathsf{v} : \sigma$ and $\Delta;\Gamma \vdash \mathsf{v} : \sigma \twoheadrightarrow \mathsf{v}'$. We need to show that $\Delta;\Gamma \vdash \mathsf{v}' \approx^{log}_{\mathcal{V}^{\mathsf{U}}} \mathsf{v} : \sigma$. Every case follows by the same reasoning as in the proof of Theorem 7.38.

Case $\Delta;\Gamma \vdash \mathsf{e} : \sigma$ and $\Delta;\Gamma \vdash \mathsf{e} : \sigma \twoheadrightarrow \mathsf{e}'$. We need to show that $\Delta;\Gamma \vdash \mathsf{e}' \approx^{log}_{\mathcal{E}^{\mathsf{U}}} \mathsf{e} : \sigma$. Almost every case follows as in Theorem 7.38.

Case $\mathsf{e} = {}^{\sigma}\mathcal{ST}\,\mathbf{e}$ then $\mathsf{e}' = \mathsf{let}\,\mathsf{x} = \mathsf{e}_u\,\mathsf{in}\,\underline{\mathsf{PROJECT}}(\sigma)\ \mathsf{x}$. We need to show that

$$(k, \mathsf{let}\,\mathsf{x} = \gamma^{\mathsf{U}}(\mathsf{e}_u)\,\mathsf{in}\,\underline{\mathsf{PROJECT}}(\sigma)\ \mathsf{x}, {}^{\sigma}\mathcal{ST}\,\rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{e}))) \in \mathcal{E}\,[\![\sigma]\!]\,\emptyset.$$

By inductive hypothesis, $(k, \gamma^{\mathsf{U}}(\mathsf{e}_u), \rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{e}))) \in \mathcal{E}^{\mathsf{U}}[\![\sigma^{\div}]\!]\rho^{\mathsf{U}}$. Then by Lemma 9.5, it is sufficient to show that for any $j \leq k$, $(j, \mathsf{v}_u, \mathbf{r}) \in \mathcal{R}^{\mathsf{U}}[\![\sigma^{\div}]\!]\rho^{\mathsf{U}}$,

$$(j, \mathsf{let}\,\mathsf{x} = \mathsf{v}_u\,\mathsf{in}\,\underline{\mathsf{PROJECT}}(\sigma)\ \mathsf{x}, {}^{\sigma}\mathcal{ST}\,\mathbf{r}) \in \mathcal{E}\,[\![\sigma]\!]\,\emptyset.$$

The result then holds by Lemma 9.10.

Case $\Delta;\Gamma \vdash \mathbf{v} : \boldsymbol{\tau}$ and $\Delta;\Gamma \vdash^{+} \mathbf{v} : \boldsymbol{\tau} \twoheadrightarrow \mathsf{v}_u$. We need to show that $\Delta;\Gamma \vdash \mathsf{v}_u \approx^{log}_{\mathcal{V}^{\mathsf{U}}} \mathbf{v} : \boldsymbol{\tau}$. Let $\rho^{\mathsf{U}}, k, \gamma^{\mathsf{U}}$ as appropriate. Most cases follow immediately by definition.

Case $\Delta;\Gamma \vdash^{+} \boldsymbol{\lambda[\alpha](\mathbf{x}\!:\!\tau).\,\mathbf{e}} : \forall[\boldsymbol{\alpha}].\,\boldsymbol{\tau} \to \boldsymbol{\theta} \twoheadrightarrow \underline{\mathsf{LAMBDA}}(\lambda(\mathsf{x}\!:\!\mathsf{U}).\,\mathsf{e}_u)$, where $\boldsymbol{\alpha}; \mathbf{x} : \boldsymbol{\tau} \vdash^{\div} \mathbf{e} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{e}_u$. Given $\cdot \vdash \boldsymbol{\tau}', R \in \mathrm{Rel}^{\mathsf{U}}[\rho^{\mathsf{U}}(\boldsymbol{\tau}')]$, $j \leq k$, $(j, \mathsf{v}_u, \mathbf{v}) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}\prime}$ where $\rho^{\mathsf{U}\prime} = \rho^{\mathsf{U}}[\boldsymbol{\alpha} \mapsto \boldsymbol{\tau}', R]$, we need to show that $(j, \gamma^{\mathsf{U}}(\mathsf{e}_u)[\mathsf{x}_u/\mathsf{v}_u], \rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{e}))[\boldsymbol{\alpha}/\boldsymbol{\tau}'][\mathbf{x}/\mathbf{v}]) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho'$. Since $\mathbf{e}, \mathsf{e}_u$ only have $\boldsymbol{\alpha}, \mathbf{x}$ and $\mathbf{x}$ free in them, this is equivalent to showing that $(j, \mathsf{e}_u[\mathsf{x}_u/\mathsf{v}_u], \mathbf{e}[\boldsymbol{\alpha}/\boldsymbol{\tau}'][\mathbf{x}/\mathbf{v}]) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho'$. By repeated use of Lemma 9.1, this is equivalent to showing $(j, \mathsf{e}_u[\mathsf{x}_u/\mathsf{v}_u], \mathbf{e}[\boldsymbol{\alpha}/\boldsymbol{\tau}'][\mathbf{x}/\mathbf{v}]) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!](\emptyset[\boldsymbol{\alpha} \mapsto \boldsymbol{\tau}', R])$, which holds by inductive hypothesis.

Case $\Delta;\Gamma \vdash^{+} \mathbf{pack}\,(\boldsymbol{\tau}',\mathbf{v})\,\mathbf{as}\,\exists\boldsymbol{\alpha}.\,\boldsymbol{\tau} : \exists\boldsymbol{\alpha}.\,\boldsymbol{\tau} \twoheadrightarrow \mathsf{v}_u$, where $\Delta;\Gamma \vdash^{+} \mathbf{v} : \boldsymbol{\tau}[\boldsymbol{\tau}'/\boldsymbol{\alpha}] \twoheadrightarrow \mathsf{v}_u$. Choose $R = \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}']\!]\rho^{\mathsf{U}}$, which is a valid choice by Lemma 9.4. Then the result holds by inductive hypothesis and Lemma 9.2.

Case $\Delta;\Gamma \vdash^{+} \mathbf{fold}_{\boldsymbol{\mu\alpha.\tau}}\,\mathbf{v} : \boldsymbol{\mu\alpha.\,\tau} \twoheadrightarrow \underline{\mathsf{FOLD}}(\mathsf{v}_u)$, where $\Delta;\Gamma \vdash^{+} \mathbf{v} : \boldsymbol{\tau}[\boldsymbol{\mu\alpha.\,\tau}/\boldsymbol{\alpha}] \twoheadrightarrow \mathsf{v}_u$. If $k = 0$, we're done. Otherwise the result holds by inductive hypothesis.

Case $\Delta;\Gamma \vdash \mathbf{r} : \boldsymbol{\theta}$ and $\Delta;\Gamma \vdash^{\div} \mathbf{r} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{v}_u$. We need to show that $\Delta;\Gamma \vdash \mathsf{v}_u \approx^{log}_{\mathcal{R}^{\mathsf{U}}} \mathbf{r} : \boldsymbol{\theta}$. Both cases follow immediately by definition.

Case $\Delta;\Gamma \vdash \mathbf{e} : \boldsymbol{\theta}$ and $\Delta;\Gamma \vdash^{\div} \mathbf{e} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{e}_u$. We need to show that $\Delta;\Gamma \vdash \mathsf{e}_u \approx^{log}_{\mathcal{E}^{\mathsf{U}}} \mathbf{e} : \boldsymbol{\theta}$. Most cases follow immediately by definition, Lemma 9.7 and Lemma 9.6.

Case $\Delta;\Gamma \vdash^{\div} \mathcal{TS}^{\sigma}\,\mathsf{e} : \sigma^{\div} \twoheadrightarrow \mathsf{let\,x} = \mathsf{e}_u\,\mathsf{in}\,\underline{\mathsf{EMBED}}(\sigma)\ \mathsf{x}$ where $\Delta;\Gamma \vdash \mathsf{e} : \sigma \twoheadrightarrow \mathsf{e}_u$.

By inductive hypothesis, we know $(k, \gamma^{\mathsf{U}}(\mathsf{e}_u), \rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{e}))) \in \mathcal{E}^{\mathsf{U}}[\![\sigma^{\div}]\!]\rho^{\mathsf{U}}$ and we need to show that

$$(k, \mathsf{let\,x} = \gamma^{\mathsf{U}}(\mathsf{e}_u)\ \mathsf{in}\ \underline{\mathsf{EMBED}}(\sigma)\ \mathsf{x}, \mathcal{TS}^{\sigma}\,\rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathsf{e}))) \in \mathcal{E}\,[\![\sigma]\!]\,\emptyset.$$

Then the result holds by Lemma 9.5 and Lemma 9.10.

Case $\Delta;\Gamma \vdash^{\div} \mathbf{v_1}\ [\boldsymbol{\tau}']\ \mathbf{v_2} : \boldsymbol{\theta}[\boldsymbol{\tau}'/\boldsymbol{\alpha}] \twoheadrightarrow \mathsf{let\,x} = \underline{\mathsf{TOFUN}}(\mathsf{v}_1)\ \mathsf{in\,x}\ \mathsf{v}_2$ where

$\Delta;\Gamma \vdash^{+} \mathbf{v_1} : \forall[\boldsymbol{\alpha}].\,\boldsymbol{\tau} \to \boldsymbol{\theta} \twoheadrightarrow \mathsf{v}_1$, and $\Delta;\Gamma \vdash^{+} \mathbf{v_2} : \boldsymbol{\tau}[\boldsymbol{\tau}'/\boldsymbol{\alpha}] \twoheadrightarrow \mathsf{v}_2$.

By inductive hypothesis, $(k, \gamma^{\mathsf{U}}(\mathsf{v}_1), \rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{v_1}))) \in \mathcal{V}^{\mathsf{U}}[\![\forall[\boldsymbol{\alpha}].\,\boldsymbol{\tau} \to \boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$ so in particular $\gamma^{\mathsf{U}}(\mathsf{v}_1) = \underline{\mathsf{LAMBDA}}(\lambda(\mathsf{x}_u : \mathsf{U}).\,\mathsf{e})$ and $\rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{v_1})) = \boldsymbol{\lambda[\alpha](x : \tau).\,e}$. Next by Lemma 9.7, Lemma 9.6 it is sufficient to show $(k, \mathsf{e}[\mathsf{x}_u/\gamma^{\mathsf{U}}(\mathsf{v}_2)], \mathbf{e}[\boldsymbol{\alpha}/\boldsymbol{\tau}'][\mathbf{x}/\rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{v_2}))]) \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\theta}[\boldsymbol{\alpha}/\boldsymbol{\tau}']]\!]\rho^{\mathsf{U}}$. By picking $\rho^{\mathsf{U}\prime} = \rho^{\mathsf{U}}[\boldsymbol{\alpha} \mapsto \boldsymbol{\tau}', \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}']\!]\rho^{\mathsf{U}}]$, the result follows by inductive hypothesis, Lemma 9.4 and Lemma 9.2.

Case $\Delta;\Gamma \vdash^{\div} \mathbf{unpack\,(\boldsymbol{\alpha}, x) = v\,in\,e} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{let\,x} = \mathsf{v}_u\ \mathsf{in\,e}_u$ where $\Delta;\Gamma \vdash^{+} \mathbf{v} : \exists\boldsymbol{\alpha}.\,\boldsymbol{\tau} \twoheadrightarrow \mathsf{v}_u$ and $\Delta, \boldsymbol{\alpha};\Gamma, \mathbf{x} : \boldsymbol{\tau} \vdash^{\div} \mathbf{e}$

By inductive hypothesis, $\rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{v})) = \mathbf{pack\,(\boldsymbol{\tau}', v')\,as\,\exists\boldsymbol{\alpha}.\,\boldsymbol{\tau}}$ and there exists $R \in \mathrm{Rel}^{\mathsf{U}}[\rho^{\mathsf{U}}(\boldsymbol{\tau}')]$ such that $(k, \mathsf{v}'_u, \mathbf{v}') \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}\prime}$ where $\rho^{\mathsf{U}\prime} = \rho^{\mathsf{U}}[\boldsymbol{\alpha} \mapsto \boldsymbol{\tau}', R]$. Then by Lemma 9.6 and Lemma 9.7 it is sufficient to show $(k, \gamma^{\mathsf{U}}(\mathsf{e}_u)[\mathsf{v}'_u/\mathsf{x}], \rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{e}))[\boldsymbol{\tau}'/\boldsymbol{\alpha}][\mathbf{v}'/\mathbf{x}]) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}}$. Since $\boldsymbol{\alpha}$ is not free in $\boldsymbol{\tau}$, by Lemma 9.1, $\mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}} = \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\tau}]\!]\rho^{\mathsf{U}\prime}$. Then the result follows by inductive hypothesis since $\rho^{\mathsf{U}\prime} \in \mathcal{D}^{\mathsf{U}}[\![\Delta, \boldsymbol{\alpha}]\!]$ and $\gamma^{\mathsf{U}}[\mathbf{x} \mapsto \mathsf{v}'_u, \mathbf{v}'] \in \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\rho^{\mathsf{U}\prime}$ by Lemma 9.1.

Case $\Delta;\Gamma \vdash^{\div} \mathbf{unfold\,v} : \boldsymbol{\tau}[\boldsymbol{\mu\alpha.\,\tau}/\boldsymbol{\alpha}] \twoheadrightarrow \mathsf{let\,x} = \underline{\mathsf{TOFOLD}}(\mathsf{v}_u)\ \mathsf{in}\ \underline{\mathsf{RETURN}}(\mathsf{x})$ where

$\Delta;\Gamma \vdash^{+} \mathbf{v} : \mathbf{E}\,\boldsymbol{\tau}_{\mathbf{exn}}\ \boldsymbol{\mu\alpha.\,\tau} \twoheadrightarrow \mathsf{v}_u$. If $k = 0$, we're done. Otherwise $k = k' + 1$, by inductive hypothesis $\gamma^{\mathsf{U}}(\mathsf{v}_u) = \underline{\mathsf{FOLD}}(\mathsf{v}'_u)$, and $\rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{v})) = \mathbf{fold}_{\boldsymbol{\mu\alpha.\tau}}\,\mathbf{v}'$ and $(k', \mathsf{v}_u, \mathbf{v}') \in \mathcal{V}^{\mathsf{U}}[\![\boldsymbol{\tau}[\boldsymbol{\mu\alpha.\,\tau}/\boldsymbol{\alpha}]]\!]$. Next, by Lemma 9.7, $\mathsf{let\,x} = \underline{\mathsf{TOFOLD}}(\underline{\mathsf{FOLD}}(\mathsf{v}'_u))\ \mathsf{in}\ \underline{\mathsf{RETURN}}(\mathsf{x}) \longmapsto^{\geq 1} \underline{\mathsf{RETURN}}(\mathsf{v}'_u)$ and $\mathbf{unfold\,fold}_{\boldsymbol{\mu\alpha.\tau}}\,\mathbf{v}' \longmapsto \mathbf{return\ v}'$. Then the result holds by Lemma 9.6 and definition of $\mathcal{K}^{\mathsf{U}}[\![\cdot]\!]$.

Case $\Delta;\Gamma \vdash^{\div} \mathbf{handle\,e\,with\,(x_1.\,e_1)\ (x_2.\,e_2)} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{let\,x}_r = \mathsf{e\,in\,case\,x}_r\ \mathsf{of},$

$$\begin{array}{l}\mathsf{x}_1\,.\,\mathsf{e}_1\\ \mathsf{x}_2\,.\,\mathsf{e}_2\end{array}$$

where $\Delta;\Gamma \vdash^{\div} \mathbf{e} : \mathbf{E}\,\boldsymbol{\tau}_{\mathbf{exn}}\,\boldsymbol{\tau} \twoheadrightarrow \mathsf{e}$, $\Delta;\Gamma, \mathbf{x_1} : \boldsymbol{\tau} \vdash^{\div} \mathbf{e_1} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{e}_1$, and $\Delta;\Gamma, \mathbf{x_1} : \boldsymbol{\tau}_{\mathbf{exn}} \vdash^{\div} \mathbf{e_2} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{e}_2$.

By inductive hypothesis and Lemma 9.5, it is sufficient to suppose $j \leq k$, $(j, \mathsf{v}_r, \mathbf{r}) \in \mathcal{R}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$ and prove $(j, \mathsf{let\,x}_r = \mathsf{v}_r\ \mathsf{in\,case\,x}_r\ \mathsf{of}\ \begin{array}{l}\\ \mathsf{x}_1\,.\,\gamma^{\mathsf{U}}(\mathsf{e}_1)\\ \mathsf{x}_2\,.\,\gamma^{\mathsf{U}}(\mathsf{e}_2)\end{array}, \mathbf{handle\,r\,with\,(x_1.}\,\rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{e_1}))\mathbf{)\ (x_2.}\,\rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{e_2}))\mathbf{)}) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$.

There are two cases, we consider the case where $\mathsf{v}_r = \underline{\mathsf{RETURN}}(\mathsf{v}_u)$ and $\mathbf{r} = \mathbf{return\ v}$, the other case is symmetric.

By computation and Lemma 9.6, it is sufficient to show $(j, \gamma^{\mathsf{U}}(\mathsf{e}_1)[\mathsf{x}_1/\mathsf{v}_u], \rho^{\mathsf{U}}(\gamma^{\mathsf{U}}(\mathbf{e_1}))[\mathbf{x}/\mathbf{v}]) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\rho^{\mathsf{U}}$.

By inductive hypothesis it is sufficient to show that $(j, \gamma^{\mathsf{U}}[\mathbf{x_1} \mapsto \mathsf{v}_u, \mathbf{v}]) \in \mathcal{G}^{\mathsf{U}}[\![\Gamma, \mathsf{x}_1 : \boldsymbol{\tau}_1]\!]\rho^{\mathsf{U}}$, which holds by assumptions about $\mathsf{v}_u, \mathbf{v}$ and Lemma 9.3.

$\square$

Lemma 9.12 (Universal Type Equivalence and Logical Equivalence Coincide in Source Contexts)
$\cdot;\Gamma \vdash \mathsf{e}' \approx^{log}_{\mathcal{E}} \mathsf{e} : \sigma$ iff $\cdot;\Gamma \vdash \mathsf{e}' \approx^{log}_{\mathcal{E}^{\mathsf{U}}} \mathsf{e} : \sigma$.

Proof

Follows directly from $\mathcal{G}\,[\![\Gamma]\!]\,\emptyset$ iff $\mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]\emptyset$ which is direct from the definition. $\square$

Theorem 9.13 (Back Translation Preserves Equivalence)
If $\cdot;\Gamma \vdash \mathsf{e}' \approx^{log}_{\mathcal{E}} \mathsf{e} : \sigma$ *and* $\cdot;\Gamma \vdash \mathsf{e}' : \sigma \twoheadrightarrow \mathsf{e}''$, *then* $\cdot;\Gamma \vdash \mathsf{e}'' \approx^{log}_{\mathcal{E}} \mathsf{e} : \sigma$.

Proof

Direct corollary of Lemma 9.12 and Theorem 9.11. □

Lemma 9.14 (Back Translation is Identity on Source Terms)

1. *If* $\mathsf{e} \in \lambda^{\mathrm{S}}$ *and* $\cdot; \mathsf{\Gamma} \vdash \mathsf{e} : \mathsf{\sigma} \twoheadrightarrow \mathsf{e}'$ *then* $\mathsf{e} = \mathsf{e}'$.
2. *If* $\mathsf{v} \in \lambda^{\mathrm{S}}$ *and* $\cdot; \mathsf{\Gamma} \vdash \mathsf{v} : \mathsf{\sigma} \twoheadrightarrow \mathsf{e}'$ *then* $\mathsf{v} = \mathsf{v}'$.

Proof

Trivial by induction. □

Lemma 9.15 (Context Back-Translation)
If $\Delta; \Gamma \vdash \mathsf{e}_1 : \mathsf{\sigma} \twoheadrightarrow \mathsf{e}_1'$ *and* $\Delta; \Gamma \vdash \mathsf{e}_2 : \mathsf{\sigma} \twoheadrightarrow \mathsf{e}_2'$*, then if* $\Delta'; \Gamma' \vdash C[\mathsf{e}_1] : \mathsf{\sigma}' \twoheadrightarrow \mathsf{e}'$*, and* $\Delta'; \Gamma' \vdash C[\mathsf{e}_1] : \mathsf{\sigma}' \twoheadrightarrow \mathsf{e}''$*, then there exists* C *such that* $\mathsf{e}' = \mathsf{C}[\mathsf{e}_1']$ *and* $\mathsf{e}'' = \mathsf{C}[\mathsf{e}_2']$.

Proof

By induction on contexts. The construction can be realized by lifting the back-translation to contexts, adding a new rule:

$$\frac{}{\Delta; \Gamma \vdash [\cdot] : \mathsf{\sigma} \twoheadrightarrow [\cdot]}$$

□

10 Translation Correctness

10.1 Semantics Preservation

Theorem 10.1 (Type Preservation)

1. *If* $\Gamma \vdash \mathsf{v} : \sigma$ *and* $\Gamma \vdash \mathsf{v} : \sigma \rightsquigarrow_v \mathbf{v}$*, then* $\cdot; \Gamma^+ \vdash \mathbf{v} : \sigma^+$.
2. *If* $\Gamma \vdash \mathsf{e} : \sigma$ *and* $\Gamma \vdash \mathsf{e} : \sigma \rightsquigarrow_e \mathbf{e}$*, then* $\cdot; \Gamma^+ \vdash \mathbf{e} : \sigma^{\div}$.

Proof

Proved simultaneously by mutual induction on the structure of v and e. We consider only the abstraction introduction case, all others follow trivially by induction.

If $\Gamma \vdash \lambda(\mathsf{x} : \sigma).\, \mathsf{e} : \sigma \to \sigma'$, then

$$\begin{aligned}\mathbf{v} = \mathbf{pack}\,(\boldsymbol{\tau}_{\mathbf{env}}, \langle \boldsymbol{\lambda}(\mathbf{z} : \langle \boldsymbol{\tau}_{\mathbf{env}}, \sigma^+ \rangle).\ & \mathbf{let}\ \mathbf{x}_{\mathbf{env}} = \mathbf{return}_{\mathbf{0}}\ \mathbf{z.1}\ \mathbf{in} \\ & \mathbf{let}\ \mathbf{y}_{\mathbf{1}} = \mathbf{return}_{\mathbf{0}}\ \mathbf{x}_{\mathbf{env}}\mathbf{.1}\ \mathbf{in} \\ & \vdots \\ & \mathbf{let}\ \mathbf{y}_{\mathbf{n}} = \mathbf{return}_{\mathbf{0}}\ \mathbf{x}_{\mathbf{env}}\mathbf{.n}\ \mathbf{in} \\ & \mathbf{let}\ \mathbf{x} = \mathbf{return}_{\mathbf{0}}\ \mathbf{z.2}\ \mathbf{in}\ \mathbf{e} \\ & , \langle \mathbf{y_1}, \ldots, \mathbf{y_n} \rangle \rangle)\ \mathbf{as}\ \exists \boldsymbol{\alpha}.\, \langle (\langle \boldsymbol{\alpha}, \sigma^+ \rangle \to \sigma'^{\div}), \boldsymbol{\alpha} \rangle\end{aligned}$$

Where $\mathrm{fv}(\lambda(\mathsf{x} : \sigma').\, \mathsf{e}) = (\mathsf{y}_1, \ldots, \mathsf{y}_n), \Gamma(\mathsf{y}_i) = \sigma_i, \Gamma' = (\mathsf{y}_1 : \sigma_1, \ldots, \mathsf{y}_n : \sigma_n), \boldsymbol{\tau}_{\mathbf{env}} = \langle {\sigma_1}^+, \ldots, {\sigma_n}^+ \rangle$, and $\Gamma', \mathsf{x} : \sigma \vdash \mathsf{e} : \sigma' \rightsquigarrow_e \mathbf{e}$.

We need to show that $\cdot; \Gamma^+ \vdash \mathbf{v} : \exists \boldsymbol{\alpha}.\, \langle (\langle \boldsymbol{\alpha}, \sigma^+ \rangle \to \sigma'^{\div}), \boldsymbol{\alpha} \rangle$.

Applying the typing rules, this reduces to showing that $\cdot; \mathbf{x}_{\mathbf{env}} : \boldsymbol{\tau}_{\mathbf{env}}, \Gamma'^+, \mathbf{x} : \sigma^+ \vdash \mathbf{e} : \sigma'^{\div}$.

By weakening it is sufficient to show that $\cdot; \Gamma'^+, \mathbf{x} : \sigma^+ \vdash \mathbf{e} : \sigma'^{\div}$ since $\mathbf{x}_{\mathbf{env}} \notin \mathrm{fv}\mathbf{e}$.

By inductive hypothesis and the fact that $\Gamma', \mathsf{x} : \sigma \vdash \mathsf{e} : \sigma' \rightsquigarrow_e \mathbf{e}$, it is sufficient to show that $\Gamma', \mathsf{x} : \sigma \vdash \mathsf{e} : \sigma'$. Which holds by the fact that $\Gamma, \mathsf{x} : \sigma \vdash \mathsf{e} : \sigma'$ and that Γ' is a subset of Γ containing all of the free variables in e besides x. □

Lemma 10.2 (Translation Weakening)
If $\Gamma \vdash \mathsf{e} : \sigma$ *and* $\Gamma \vdash \mathsf{e} : \sigma \rightsquigarrow_e \mathbf{e}$*, then for any* $\Gamma' \subset \Gamma$ *such that* $\Gamma \vdash \mathsf{e} : \sigma$, $\Gamma' \vdash \mathsf{e} : \sigma \rightsquigarrow_e \mathbf{e}$.

Proof

By induction on e. □

Lemma 10.3 (Context Translation)
If $\vdash \mathsf{C} : (\Gamma \vdash \sigma) \Rightarrow (\Gamma' \vdash \sigma')$, $\Gamma \vdash \mathsf{e} : \sigma$ *and* $\Gamma \vdash \mathsf{e} : \sigma \rightsquigarrow_e \mathbf{e}$*, then there exists* $\mathbf{C}$ *such that* $\Gamma' \vdash \mathsf{C}[\mathsf{e}] : \sigma \rightsquigarrow_e \mathbf{C}[\mathbf{e}]$.
Furthermore if $\Gamma \vdash \mathsf{e}' : \sigma$ *and* $\Gamma \vdash \mathsf{e}' : \sigma \rightsquigarrow_e \mathbf{e}'$*, then* $\Gamma' \vdash \mathsf{C}[\mathsf{e}] : \sigma \rightsquigarrow_e \mathbf{C}[\mathbf{e}']$.

Proof

Both follow by induction on C, using Lemma 10.2 in the abstraction case. □

Lemma 10.4 (Boundary Terminates (Source to Target))
If $\cdot \vdash$
If $\Delta; \cdot \vdash \mathsf{v} : \sigma$*, then there exist* $n, \mathbf{v}$ *such that* $\mathcal{TS}^{\sigma}\, \mathsf{v} \longmapsto^n \mathbf{return_0}\ \mathbf{v}$.

Proof

By induction on the typing derivation. We omit the cases for unit, sums, and pairs.

Case $\Delta; \cdot \vdash \mathsf{v} : \sigma_1 \to \sigma_2$: Then $\mathcal{TS}^{\sigma_1 \to \sigma_2}\, \mathsf{v} \longmapsto$

$$\boldsymbol{\lambda(z : \langle 1, {\sigma_1}^{+}\rangle).}$$
$$\mathcal{TS}^{\sigma_2}\begin{pmatrix}\mathsf{let\ a} = {}^{\sigma_1}\mathcal{ST}\ \mathbf{return_0\ z.2}\ \mathsf{in} \\ \quad \mathsf{v\ a}\end{pmatrix}$$

Case $\Delta; \cdot \vdash \mathsf{fold}_{\mu\alpha.\sigma'}\ \mathsf{v}' : \mu\alpha.\,\sigma'$:

First, $\mathcal{TS}^{\mu\alpha.\sigma'}\ \mathsf{fold}_{\mu\alpha.\sigma'}\ \mathsf{v}' \longmapsto^2$ $\mathbf{let\ v} = \mathcal{TS}^{\sigma[\mu\alpha.\sigma/\alpha]}\ \mathsf{v}'\ \mathbf{in\ return_0\ fold_{\mu\alpha.\tau}\ v}$. By inductive hypothesis there exist $n, \mathbf{v}'$ such that $\mathcal{TS}^{\sigma[\mu\alpha.\sigma/\alpha]}\ \mathsf{v}' \longmapsto^n \mathbf{return_0\ v'}$
Then by definition of the operational semantics, $\mathcal{TS}^{\mu\alpha.\sigma'}\ \mathsf{fold}_{\mu\alpha.\sigma'}\ \mathsf{v}' \longmapsto^{n+3} \mathbf{return_0\ fold_{\mu\alpha.\tau}\ v'}$.

□

Lemma 10.5 **(Boundary Terminates (Target to Source))**
If $\Delta; \cdot \vdash \mathbf{v} : \sigma^+$*, then there exist* n, v *such that* ${}^{\sigma}\mathcal{ST}\ \mathbf{return\ v} \longmapsto^n \mathsf{v}$*.*

Proof

By induction on the typing derivation. We omit the cases for sums and tuples.

Case $\Delta; \cdot \vdash \mathbf{v} : \sigma_1 \to {\sigma_2}^+$: Then ${}^{\sigma_1 \to \sigma_2}\mathcal{ST}\ \mathbf{return\ v} \longmapsto$

$$\lambda(\mathsf{x} : \sigma_1).\ {}^{\sigma_2}\mathcal{ST}\begin{pmatrix}\mathbf{unpack}\ (\boldsymbol{\alpha}, \mathbf{z}) = \mathbf{v\ in\ let\ x_f} = \mathbf{return_0\ z.1\ in} \\ \qquad\qquad\qquad\qquad \mathbf{let\ x_{env}} = \mathbf{return_0\ z.2\ in} \\ \qquad\qquad\qquad\qquad \mathbf{let\ x} = \mathcal{TS}^{\sigma_1}\ \mathsf{x}\ \mathbf{in\ x_f}\ [\boldsymbol{\alpha}]\ \langle \mathbf{x_{env}}, \mathbf{x}\rangle\end{pmatrix}$$

Case $\Delta; \cdot \vdash \mathbf{fold}_{\mu\alpha.\sigma'^+}\ \mathbf{v}' : \boldsymbol{\mu\alpha.}\ {\sigma'}^+$: Directly analogous to the case in Lemma 10.4.

□

Lemma 10.6 **(Boundary Cancellation (Source round-trip))**
If $\rho \in \mathcal{D}[\![\Delta]\!]$ *and* $\cdot \vdash \sigma$*, then*

1. *If* $(k, \mathsf{e}_1, \mathsf{e}_2) \in \mathcal{E}[\![\sigma]\!]\,\rho$ *then* $(k, \mathsf{e}_1, {}^{\sigma}\mathcal{ST}\,\mathcal{TS}^{\sigma}\ \mathsf{e}_2) \in \mathcal{E}[\![\sigma]\!]\,\rho$
2. *If* $(k, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\sigma]\!]\,\rho$ *and* ${}^{\sigma}\mathcal{ST}\,\mathcal{TS}^{\sigma}\ \mathsf{v}_2 \longmapsto^n \mathsf{v}_2'$ *then* $(k, \mathsf{v}_1, \mathsf{v}_2') \in \mathcal{V}[\![\sigma]\!]\,\rho$

Proof

Proved simultaneously by induction on k and σ. We omit the cases for unit, sums, and pairs.

1. By Lemma 7.9, it is sufficient to prove that for every $j \le k$, if $(j, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\sigma]\!]\,\rho$ then $(j, \mathsf{v}_1, {}^{\sigma}\mathcal{ST}\,\mathcal{TS}^{\sigma}\ \mathsf{v}_2) \in \mathcal{E}[\![\sigma]\!]\,\rho$. Then by Lemma 10.5, Lemma 10.4, ${}^{\sigma}\mathcal{ST}\,\mathcal{TS}^{\sigma}\ \mathsf{v}_2 \longmapsto^n \mathsf{v}_2'$ for some n, v_2', so the result holds by inductive hypothesis, Lemma 7.11 and Lemma 7.8.
2. Values

   Case $\sigma = \sigma_1 \to \sigma_2$:
   By definition of $\mathcal{V}[\![\sigma_1 \to \sigma_2]\!]\,\rho$, $\mathsf{v}_1 = \lambda(\mathsf{x} : \sigma_1).\,\mathsf{e}_1$ and $\mathsf{v}_2 = \lambda(\mathsf{x} : \sigma_1).\,\mathsf{e}_2$ where for every $j \le k$, $(j, \mathsf{v}_1'', \mathsf{v}_2'') \in \mathcal{V}[\![\sigma_1]\!]\,\rho$, $(j, \mathsf{e}_1[\mathsf{v}_1''/\mathsf{x}], \mathsf{e}_2[\mathsf{v}_2''/\mathsf{x}]) \in \mathcal{E}[\![\sigma_2]\!]\,\rho$.
   Then, as in Lemma 10.4 and Lemma 10.5,
   ${}^{\sigma}\mathcal{ST}\,\mathcal{TS}^{\sigma}\ \mathsf{v}_2 \longmapsto$
   $${}^{\sigma}\mathcal{ST}\ \mathbf{return_0\ pack}\ (\mathbf{1}, \langle \boldsymbol{\lambda(z : \langle 1, {\sigma_1}^{+}\rangle).}\ \mathcal{TS}^{\sigma_2}\begin{pmatrix}\mathsf{let\ a} = {}^{\sigma_1}\mathcal{ST}\ \mathbf{return_0\ z.2}\ \mathsf{in} \\ \quad \mathsf{v_2\ a}\end{pmatrix}, \langle\rangle\rangle)\ \mathbf{as}\ (\sigma_1 \to \sigma_2)^+$$

$\longmapsto$

$$\lambda(\mathsf{y}:\sigma_1).\,{}^{\sigma_2}\mathcal{ST}\left(\begin{array}{l}\textbf{unpack}\,(\alpha,\mathbf{w}) = \textbf{pack}\,(\mathbf{1},\langle\lambda(\mathbf{z}:\langle\mathbf{1},\sigma_1{}^{+}\rangle).\quad\quad, \langle\rangle\rangle)\,\textbf{as}\,(\sigma_1\to\sigma_2)^{+}\,\textbf{in}\\ \quad\quad\quad\quad\quad\quad\mathcal{TS}^{\,\sigma_2}\left(\begin{array}{l}\mathsf{let\,a} = {}^{\sigma_1}\mathcal{ST}\,\textbf{return}_{\mathbf{0}}\ \mathbf{z.2}\,\mathsf{in}\\ \quad \mathsf{v}_2\ \mathsf{a}\end{array}\right)\\ \quad\textbf{let}\,\mathbf{x_f} = \textbf{return}_{\mathbf{0}}\ \mathbf{w.1}\,\textbf{in}\\ \quad\textbf{let}\,\mathbf{x_{env}} = \mathbf{w.2}\,\textbf{in}\\ \quad\textbf{let}\,\mathbf{x} = \mathcal{TS}^{\,\sigma_1}\,\mathsf{y}\,\textbf{in}\,\mathbf{x_f}\ [\alpha]\ \langle\mathbf{x_{env}},\mathbf{x}\rangle\end{array}\right)$$

Then this is v_2', so we need to show that $(k, \mathsf{v}_1, \mathsf{v}_2') \in \mathcal{V}[\![\sigma_1 \to \sigma_2]\!]\,\rho$.
Suppose $j \leq k$, $(j, \mathsf{v}_1'', \mathsf{v}_1'') \in \mathcal{V}[\![\sigma_1]\!]\,\rho$. We need to show
$(j, \mathsf{e}_1[\mathsf{v}_1''/\mathsf{x}],$

$${}^{\sigma_2}\mathcal{ST}\left(\begin{array}{l}\textbf{unpack}\,(\alpha,\mathbf{w}) = \textbf{pack}\,(\mathbf{1},\langle\lambda(\mathbf{z}:\langle\mathbf{1},\sigma_1{}^{+}\rangle).\quad\quad, \langle\rangle\rangle)\,\textbf{as}\,(\sigma_1\to\sigma_2)^{+}\,\textbf{in}\\ \quad\quad\quad\quad\quad\quad\mathcal{TS}^{\,\sigma_2}\left(\begin{array}{l}\mathsf{let\,a} = {}^{\sigma_1}\mathcal{ST}\,\textbf{return}_{\mathbf{0}}\ \mathbf{z.2}\,\mathsf{in}\\ \quad \mathsf{v}_2\ \mathsf{a}\end{array}\right)\\ \quad\textbf{let}\,\mathbf{x_f} = \textbf{return}_{\mathbf{0}}\ \mathbf{w.1}\,\textbf{in}\\ \quad\textbf{let}\,\mathbf{x_{env}} = \mathbf{w.2}\,\textbf{in}\\ \quad\textbf{let}\,\mathbf{x} = \mathcal{TS}^{\,\sigma_1}\,\mathsf{v}_2''\,\textbf{in}\,\mathbf{x_f}\ [\alpha]\ \langle\mathbf{x_{env}},\mathbf{x}\rangle\end{array}\right)) \in$$

$\mathcal{V}[\![\sigma_2]\!]\,\rho$.

First, $${}^{\sigma_2}\mathcal{ST}\left(\begin{array}{l}\textbf{unpack}\,(\alpha,\mathbf{w}) = \textbf{pack}\,(\mathbf{1},\langle\lambda(\mathbf{z}:\langle\mathbf{1},\sigma_1{}^{+}\rangle).\quad\quad, \langle\rangle\rangle)\,\textbf{as}\,(\sigma_1\to\sigma_2)^{+}\,\textbf{in}\\ \quad\quad\quad\quad\quad\quad\mathcal{TS}^{\,\sigma_2}\left(\begin{array}{l}\mathsf{let\,a} = {}^{\sigma_1}\mathcal{ST}\,\textbf{return}_{\mathbf{0}}\ \mathbf{z.2}\,\mathsf{in}\\ \quad \mathsf{v}_2\ \mathsf{a}\end{array}\right)\\ \quad\textbf{let}\,\mathbf{x_f} = \textbf{return}_{\mathbf{0}}\ \mathbf{w.1}\,\textbf{in}\\ \quad\textbf{let}\,\mathbf{x_{env}} = \mathbf{w.2}\,\textbf{in}\\ \quad\textbf{let}\,\mathbf{x} = \mathcal{TS}^{\,\sigma_1}\,\mathsf{v}_2''\,\textbf{in}\,\mathbf{x_f}\ [\alpha]\ \langle\mathbf{x_{env}},\mathbf{x}\rangle\end{array}\right)$$

$\longmapsto^5$

$${}^{\sigma_2}\mathcal{ST}\,\textbf{let}\,\mathbf{x} = \mathcal{TS}^{\,\sigma_1}\,\mathsf{v}_2'\,\textbf{in}\left(\begin{array}{l}\lambda(\mathbf{z}:\langle\mathbf{1},\sigma_1{}^{+}\rangle).\\ \quad\mathcal{TS}^{\,\sigma_2}\left(\begin{array}{l}\mathsf{let\,a} = {}^{\sigma_1}\mathcal{ST}\,\textbf{return}_{\mathbf{0}}\ \mathbf{z.2}\,\mathsf{in}\\ \quad \mathsf{v}_2\ \mathsf{a}\end{array}\right)\end{array}\right)[\alpha]\ \langle\langle\rangle,\mathbf{x}\rangle.$$

By Lemma 10.4, $\mathcal{TS}^{\sigma_1}\, v_2''' \longmapsto^n \mathbf{v_2'''}$ for some $n, \mathbf{v_2'''}$. Then,

$$^{\sigma_2}\mathcal{ST}\ \mathbf{let}\ \mathbf{x} = \mathcal{TS}^{\sigma_1}\, v_2'\ \mathbf{in}\ \left(\begin{array}{l} \boldsymbol{\lambda}(\mathbf{z} : \langle \mathbf{1}, \sigma_1{}^{+} \rangle). \\ \quad \mathcal{TS}^{\sigma_2} \left(\begin{array}{l} \mathsf{let}\ \mathsf{a} = {}^{\sigma_1}\mathcal{ST}\ \mathbf{return_0}\ \mathbf{z.2}\ \mathsf{in} \\ \quad \mathsf{v_2}\ \mathsf{a} \end{array} \right) \end{array} \right) [\boldsymbol{\alpha}]\ \langle \langle \rangle, \mathbf{x} \rangle$$

$\longmapsto^4\ {}^{\sigma_2}\mathcal{ST}\,\mathcal{TS}^{\sigma_2} \left(\begin{array}{l} \mathsf{let}\ \mathsf{a} = {}^{\sigma_1}\mathcal{ST}\ \mathbf{return_0}\ \mathbf{v_2'''}\ \mathsf{in} \\ \quad \mathsf{v_2}\ \mathsf{a} \end{array} \right)$ By Lemma 10.5, ${}^{\sigma_1}\mathcal{ST}\ \mathbf{return_0}\ \mathbf{v_2'''} \longmapsto^m v_2''''$ for some m, v_2''''. Note that this means ${}^{\sigma_1}\mathcal{ST}\,\mathcal{TS}^{\sigma_1}\, v_2'' \longmapsto^{m+n} v_2''''$, so by inductive hypothesis, $(j, v_1'', v_2'''') \in \mathcal{V}[\![\sigma_1]\!]\,\rho$. Finally,

${}^{\sigma_2}\mathcal{ST}\,\mathcal{TS}^{\sigma_2} \left(\begin{array}{l} \mathsf{let}\ \mathsf{a} = {}^{\sigma_1}\mathcal{ST}\ \mathbf{return_0}\ \mathbf{v_2'''}\ \mathsf{in} \\ \quad \mathsf{v_2}\ \mathsf{a} \end{array} \right) \longmapsto^{m+2}\ {}^{\sigma_2}\mathcal{ST}\,\mathcal{TS}^{\sigma_2}\, e_2[v_2''''/x]$, so by Lemma 7.11, it is sufficient to prove $(j, e_1[v_1''/x], {}^{\sigma_2}\mathcal{ST}\,\mathcal{TS}^{\sigma_2}\, e_2[v_2''''/x])$ which holds by inductive hypothesis and the fact that $(j, e_1[v_1''/x], e_2[v_2''x/)] \in \mathcal{E}[\![\sigma_2]\!]\,\rho$.

Case $\sigma = \mu\alpha.\,\sigma'$: By definition of $\mathcal{V}[\![\mu\alpha.\,\sigma']\!]\,\rho$, $v_1 = \mathsf{fold}_{\mu\alpha.\sigma'}\, v_1'$ and $v_2 = \mathsf{fold}_{\mu\alpha.\sigma'}\, v_2'$. where $(k-1, v_1', v_2') \in \mathcal{V}[\![\sigma'[\mu\alpha.\,\sigma'/\alpha]]\!]\,\rho$.
Next as in the proof of Lemma 10.4, ${}^{\mu\alpha.\sigma'}\mathcal{ST}\,\mathcal{TS}^{\mu\alpha.\sigma'}\, 3 \longmapsto^{n+3}\ {}^{\mu\alpha.\sigma'}\mathcal{ST}\ \mathbf{return_0}\ \mathbf{fold}_{\boldsymbol{\mu\alpha.\sigma'^{+}}}\ \mathbf{v_2''}$ where $\mathcal{TS}^{\sigma'[\mu\alpha.\sigma'/\alpha]}\, v_2'' \longmapsto^n \mathbf{return}\ \mathbf{v_2''}$. Then as in the proof of Lemma 10.5, ${}^{\mu\alpha.\sigma'}\mathcal{ST}\ \mathbf{return_0}\ \mathbf{fold}_{\boldsymbol{\mu\alpha.\sigma'^{+}}}\ \mathbf{v_2''} \longmapsto^{m+3} \mathsf{fold}_{\mu\alpha.\sigma'}\, v_2'''$ where ${}^{\sigma'[\mu\alpha.\sigma'/\alpha]}\mathcal{ST}\ \mathbf{return}\ \mathbf{v_2''} \longmapsto^m v_2'''$. Then ${}^{\sigma'[\mu\alpha.\sigma'/\alpha]}\mathcal{ST}\,\mathcal{TS}^{\sigma'[\mu\alpha.\sigma'/\alpha]}\, v_2'' \longmapsto^{m+n} v_2'''$, so by inductive hypothesis $(k-1, v_1'', v_2''') \in \mathcal{V}[\![\sigma'[\mu\alpha.\,\sigma'/\alpha]]\!]\,\rho$.

□

Lemma 10.7 (Boundary Cancellation (Target round-trip))
If $\rho \in \mathcal{D}[\![\Delta]\!]$ and $\cdot \vdash \sigma$, then

1. *If $(k, \mathbf{e_1}, \mathbf{e_2}) \in \mathcal{E}[\![\sigma^{\div}]\!]\,\rho$ then $(k, \mathbf{e_1}, \mathcal{TS}^{\sigma}\ {}^{\sigma}\mathcal{ST}\ \mathbf{e_2}) \in \mathcal{E}[\![\sigma^{\div}]\!]\,\rho$*
2. *If $(k, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\sigma^{+}]\!]\,\rho$ and $\mathcal{TS}^{\sigma}\ {}^{\sigma}\mathcal{ST}\ \mathbf{return}\ \mathbf{v_2} \longmapsto^n \mathbf{return}\ \mathbf{v_2'}$ then $(k, \mathbf{v_1}, \mathbf{v_2'}) \in \mathcal{V}[\![\sigma^{+}]\!]\,\rho$*

Proof

1. Applying Lemma 7.9 there are two cases.

   Case Suppose $j \leq k$ and $(j, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\sigma^{+}]\!]\,\rho$. We need to show that $(j, \mathbf{return}\ \mathbf{v_1}, \mathcal{TS}^{\sigma}\ {}^{\sigma}\mathcal{ST}\ \mathbf{return}\ \mathbf{v_2}) \in \mathcal{E}[\![\sigma^{\div}]\!]\,\rho$.
   By Lemma 10.5 and Lemma 10.4, there exist $n, \mathbf{v_2'}$ such that $\mathcal{TS}^{\sigma}\ {}^{\sigma}\mathcal{ST}\ \mathbf{return}\ \mathbf{v_2} \longmapsto^n \mathbf{return}\ \mathbf{v_2'}$. By Lemma 7.11, it is sufficient to show that $(j, \mathbf{return}\ \mathbf{v_1}, \mathbf{return}\ \mathbf{v_2'}) \in \mathcal{E}[\![\sigma^{\div}]\!]\,\rho$, which holds by Lemma 7.8 and part 2.

   Case Suppose $j \leq k$ and $(j, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\mathbf{0}]\!]\,\rho$. We need to show that $(j, \mathbf{raise}\ \mathbf{v_1}, \mathcal{TS}^{\sigma}\ {}^{\sigma}\mathcal{ST}\ \mathbf{raise}\ \mathbf{v_2}) \in \mathcal{E}[\![\sigma^{\div}]\!]\,\rho$. This holds vacuously since $\mathcal{V}[\![\mathbf{0}]\!]\,\rho = \emptyset$.

2. Values We omit the cases for unit, sums, and pairs.

   Case $\sigma = \sigma_1 \to \sigma_2, (\sigma_1 \to \sigma_2)^{+} = \boldsymbol{\exists\alpha.}\ \langle (\langle \boldsymbol{\alpha}, \sigma_1{}^{+} \rangle \to \sigma_2{}^{\div}), \boldsymbol{\alpha} \rangle$:
   By definition of $\mathcal{V}\left[\!\!\left[(\sigma_1 \to \sigma_2)^{+}\right]\!\!\right]\rho$, $\mathbf{v_1} = \mathbf{pack}\ (\boldsymbol{\tau_1}, \langle \mathbf{v_1''}, \mathbf{v_1'''} \rangle)\ \mathbf{as}\ (\sigma_1 \to \sigma_2)^{+}$ and $\mathbf{v_2} = \mathbf{pack}\ (\boldsymbol{\tau_2}, \langle \mathbf{v_2''}, \mathbf{v_2'''} \rangle)\ \mathbf{as}\ (\sigma$ such that there exists $R \in \mathrm{Rel}[\boldsymbol{\tau_1}, \boldsymbol{\tau_2}]$ such that $(k, \mathbf{v_1'''}, \mathbf{v_2'''}) \in \mathcal{V}[\![\boldsymbol{\alpha}]\!]\,\rho'$ and $(k, \mathbf{v_1''}, \mathbf{v_2''}) \in \mathcal{V}[\![\langle \boldsymbol{\alpha}, \sigma_1{}^{+} \rangle \to \sigma_2{}^{\div}]\!]\,\rho'$ where $\rho' = \rho[\boldsymbol{\alpha} \mapsto (\boldsymbol{\tau_1}, \boldsymbol{\tau_2}, R)]$.

Furthermore, $\mathbf{v}_1'' = \lambda(\mathbf{x} : \langle \boldsymbol{\tau}_1, \sigma_1{}^+\rangle).\, \mathbf{e}_1$ and $\mathbf{v}_2'' = \lambda(\mathbf{x} : \langle \boldsymbol{\tau}_2, \sigma_1{}^+\rangle).\, \mathbf{e}_2$ such that for any $j \leq k, (j, \mathbf{v}_1''''', \mathbf{v}_2''''') \in \mathcal{V}\,[\![\langle \alpha, \sigma_1{}^+\rangle]\!]\,\rho'$, $(j, \mathbf{e}_1[\mathbf{v}_1'''''/\mathbf{x}], \mathbf{e}_2[\mathbf{v}_2'''''\mathbf{x}/)] \in \mathcal{E}\,[\![\sigma_2{}^\div]\!]\,\rho'$.
Next, as in the proofs of Lemma 10.5 and Lemma 10.4,
${}^{\sigma_1 \to \sigma_2}\mathcal{ST}\,\textbf{return}\ \mathbf{v_2} \longmapsto$
$\lambda(\mathsf{x} : \sigma_1).\ {}^{\sigma_2}\mathcal{ST}\,(\textbf{unpack}\,(\alpha, \mathbf{z}) = \mathbf{v_2}\ \textbf{in}\ \textbf{let}\ \mathbf{x_f} = \textbf{return}\ \mathbf{z.1}\ \textbf{in}\ \textbf{let}\ \mathbf{x_{env}} = \textbf{return}\ \mathbf{z.2}\ \textbf{in}\ \textbf{let}\ \mathbf{x} = \mathcal{TS}^{\,\sigma_1}\,\mathsf{x}\ \textbf{in}\ \mathbf{x_f}\ [\alpha]\ \langle \mathbf{x_{env}}, \mathbf{x}\rangle)$ which we denote v_2'
and $\mathcal{TS}^{\,\sigma_1 \to \sigma_2}\,\mathsf{v}_2' \longmapsto$
$\textbf{return pack}\,(\mathbf{1}, \langle \lambda(\mathbf{z} : \langle \mathbf{1}, \sigma''^+\rangle).\ \mathcal{TS}^{\,\sigma'}\begin{pmatrix}\mathsf{let\ x} = {}^{\sigma''}\mathcal{ST}\,\textbf{return}_\mathbf{0}\ \mathbf{z.2}\ \mathsf{in} \\ \mathsf{v}_2'\ \mathsf{x}\end{pmatrix}, \langle\rangle\rangle)\ \textbf{as}\,(\sigma_1 \to \sigma_2)^+$ and we define the value in the **return** here to be $\mathbf{v}_2'$. Then $\mathcal{TS}^{\,\sigma_1 \to \sigma_2}\ {}^{\sigma_1 \to \sigma_2}\mathcal{ST}\,\textbf{return}\ \mathbf{v_2} \longmapsto^2 \textbf{return}\ \mathbf{v}_2'$, so we need to show that $(k, \mathbf{v_1}, \mathbf{v}_2') \in \mathcal{V}\,[\![\exists \alpha.\, \langle (\langle \alpha, \sigma_1{}^+\rangle \to \sigma_2{}^\div), \alpha\rangle]\!]\,\rho$.
We define R' to be the relation $\{(j, \mathbf{v}_1''', \langle\rangle) | j \leq k\}$. Then $R' \in \mathrm{Rel}[\boldsymbol{\tau}_1, \mathbf{1}]$. Define $\rho'' = \rho[\alpha \mapsto (\boldsymbol{\tau}_1, \mathbf{1}, R')]$. Then $(k, \mathbf{v}_1''', \langle\rangle) \in \mathcal{V}\,[\![\alpha]\!]\,\rho''$.
Next, we need to show that for every $(j, \mathbf{v^2}_1, \mathbf{v^2}_2) \in \mathcal{V}\,[\![\langle \alpha, \sigma_1{}^+\rangle]\!]\,\rho''$,
$(j, \mathbf{e_1}[\mathbf{v^2}_1/\mathbf{z}], \mathcal{TS}^{\,\sigma'}\begin{pmatrix}\mathsf{let\ x} = {}^{\sigma''}\mathcal{ST}\,\textbf{return}_\mathbf{0}\ (\mathbf{v^2}_2)\mathbf{.2}\ \mathsf{in} \\ \mathsf{v}_2'\ \mathsf{x}\end{pmatrix}) \in \mathcal{E}\,[\![\sigma_2{}^\div]\!]\,\rho''$. By definition of $\mathcal{V}\,[\![\langle \alpha, \sigma_1{}^+\rangle]\!]\,\rho''$, $\mathbf{v^2}_1 = \langle \mathbf{v}_1''', \mathbf{v^3}_1\rangle, \mathbf{v^2}_2 = \langle\langle\rangle, \mathbf{v^3}_2\rangle$ such that $(j, \mathbf{v^3}_1, \mathbf{v^3}_2) \in \mathcal{V}\,[\![\sigma_1{}^+]\!]\,\rho''$.
Furthermore by Lemma 10.5 and Lemma 10.4, there exist $m, n, \mathsf{v^3}_2, \mathbf{v^3}_2$ such that ${}^{\sigma_1}\mathcal{ST}\,\textbf{return}\ \mathbf{v^3}_2 \longmapsto^m \mathsf{v^4}_2$ and $\mathcal{TS}^{\,\sigma_1}\,\mathsf{v^4}_2 \longmapsto^n \textbf{return}\ \mathbf{v^4}_2$.
$\mathcal{TS}^{\,\sigma'}\begin{pmatrix}\mathsf{let\ x} = {}^{\sigma''}\mathcal{ST}\,\textbf{return}_\mathbf{0}\ (\mathbf{v^2}_2)\mathbf{.2}\ \mathsf{in} \\ \mathsf{v}_2'\ \mathsf{x}\end{pmatrix} \longmapsto^{m+2} \mathbf{v}_2'\ \mathsf{v^4}_2 \longmapsto^{n+8} \mathbf{e_2}[\langle \mathbf{v}_2''', \mathbf{v^4}_2\rangle/\mathbf{x}]$.
Thus by Lemma 7.11 and fact that $(k, \mathbf{v}_1'', \mathbf{v}_2'') \in \mathcal{V}\,[\![\langle \alpha, \sigma_1{}^+\rangle \to \sigma_2{}^\div]\!]\,\rho'$ it is sufficient to show that $(j, \mathbf{v^2}_1, \langle \mathbf{v}_2''', \mathbf{v^4}_2\rangle) \in \mathcal{V}\,[\![\langle \alpha, \sigma_1{}^+\rangle]\!]\,\rho'$.
Recalling that $\mathbf{v^2}_1 = \langle \mathbf{v}_1''', \mathbf{v^3}_1\rangle$, we get $(j, \mathbf{v}_1''', \mathbf{v}_2''') \in \mathcal{V}\,[\![\alpha]\!]\,\rho'$ by Lemma 7.6. Finally we need to show that $(j, \mathbf{v^3}_1, \mathbf{v^4}_2) \in \mathcal{V}\,[\![\sigma_1{}^+]\!]\,\rho'$. By inductive hypothesis and the fact that $(j, \mathbf{v^3}_1, \mathbf{v^3}_2) \in \mathcal{V}\,[\![\sigma^+]\!]\,\rho''$, $(j, \mathbf{v^3}_1, \mathbf{v^4}_2) \in \mathcal{V}\,[\![\sigma^+]\!]\,\rho''$. Then the property holds by two applications of Lemma 7.4

Case $\sigma = \mu\alpha.\,\sigma', (\mu\alpha.\,\sigma')^+ = \boldsymbol{\mu\alpha.}\,\sigma'^+$: the proof is directly analogous to the case in Lemma 10.6.

□

Lemma 10.8 (Boundary Cancellation Equivalence)

1. *If* $\Delta; \Gamma \vdash \mathsf{e} : \sigma$, *then* $\Delta; \Gamma \vdash \mathsf{e} \approx^{ctx}_{\mathrm{ST}} {}^{\sigma}\mathcal{ST}\,(\mathcal{TS}^{\,\sigma}\,\mathsf{e}) : \sigma$.
2. *If* $\Delta; \Gamma \vdash \mathbf{e} : \sigma^\div$, *then* $\Delta; \Gamma \vdash \mathbf{e} \approx^{ctx}_{\mathrm{ST}} \mathcal{TS}^{\,\sigma}\,({}^{\sigma}\mathcal{ST}\,\mathbf{e}) : \sigma^\div$.

Proof

By Theorem 7.43, induction on the step index, Lemma 10.6 and Lemma 10.7. □

Lemma 10.9 (Cross Language Relation Alternative)

1. $(k, \mathsf{e}, \mathbf{e}) \in \mathcal{E}^\div[\![\sigma]\!]iff(k, \mathsf{e}, {}^{\sigma}\mathcal{ST}\,\mathbf{e}) \in \mathcal{E}\,[\![\sigma]\!]\,\emptyset$
2. $(k, \mathsf{e}, \mathbf{e}) \in \mathcal{E}^\div[\![\sigma]\!]iff(k, \mathcal{TS}^{\,\sigma}\,\mathsf{e}, \mathbf{e}) \in \mathcal{E}\,[\![\sigma^\div]\!]\,\emptyset$
3. $\cdot \vdash \mathsf{e} \approx_\div \mathbf{e} : \sigma$iff$\cdot; \cdot \vdash \mathsf{e} \approx^{ctx}_{\mathrm{ST}} {}^{\sigma}\mathcal{ST}\,\mathbf{e} : \sigma$

4. $\cdot \vdash \mathsf{e} \approx_{\div} \mathbf{e} : \sigma \text{iff} \cdot ; \cdot \vdash \mathcal{TS}^{\sigma}\, \mathsf{e} \approx^{ctx}_{\mathrm{ST}} \mathbf{e} : \sigma^{\div}$

Proof

Case Expansion of definition.

Case By previous case, Lemma 7.35 and Lemma 10.7.

Case By above and induction on k.

Case By above and induction on k.

□

Lemma 10.10 (Contextual Boundary Cancellation)

1. $(k, e_1, C[\mathsf{v_2}]) \in \mathcal{E}\,[\![\theta]\!]\,\rho$ iff $(k, e_1, C[\mathsf{v}'_2]) \in \mathcal{E}\,[\![\theta]\!]\,\rho$ *where* ${}^{\sigma}\mathcal{ST}\,\mathcal{TS}^{\sigma}\,\mathsf{v_2} \longmapsto^{*} \mathsf{v}'_2$.
2. $(k, e_1, C[\mathsf{e_2}]) \in \mathcal{E}\,[\![\theta]\!]\,\rho$ iff $(k, e_1, C[{}^{\sigma}\mathcal{ST}\,\mathcal{TS}^{\sigma}\,\mathsf{e_2}]) \in \mathcal{E}\,[\![\theta]\!]\,\rho$
3. $(k, e_1, C[\mathbf{v_2}]) \in \mathcal{E}\,[\![\theta]\!]\,\rho$ iff $(k, e_1, C[\mathbf{v}'_2]) \in \mathcal{E}\,[\![\theta]\!]\,\rho$ *where* $\mathcal{TS}^{\sigma}\,{}^{\sigma}\mathcal{ST}\,\mathbf{return_0\ v_2} \longmapsto^{*} \mathbf{return_0\ v'_2}$.
4. $(k, e_1, C[\mathbf{e_2}]) \in \mathcal{E}\,[\![\theta]\!]\,\rho$ iff $(k, e_1, C[\mathcal{TS}^{\sigma}\,{}^{\sigma}\mathcal{ST}\,\mathbf{e_2}]) \in \mathcal{E}\,[\![\theta]\!]\,\rho$

Proof

By Lemma 7.39, Lemma 10.6 and Lemma 10.7. □

Lemma 10.11 (Cross-Language Monadic Bind)
If $(k, \mathsf{e}, \mathbf{e}) \in \mathcal{E}^{\div}[\![\sigma]\!]$ *and for all* $j \le k$, *if* $(j, \mathsf{v}, \mathbf{v}) \in \mathcal{V}^{+}[\![\sigma]\!]$ *then* $(j, \mathsf{K}[\mathsf{v}], \mathbf{K}[\mathbf{return_0\ v}]) \in \mathcal{E}^{\div}[\![\sigma']\!]$, *then* $(k, \mathsf{K}[\mathsf{e}], \mathbf{K}[\mathbf{e}]) \in \mathcal{E}^{\div}[\![\sigma']\!]$.

Proof

Applying Lemma 10.10 and definition of $\mathcal{E}^{\div}[\![\sigma']\!]$, it is sufficient to prove that $(k, \mathsf{K}[\mathsf{e}], {}^{\sigma'}\mathcal{ST}\,\mathbf{K}[\mathcal{TS}^{\sigma}\,{}^{\sigma}\mathcal{ST}\,\mathbf{e}]) \in \mathcal{E}\,[\![\sigma']\!]\,\emptyset$.

By Lemma 7.9, it is sufficient to prove that for all $j \le k$ and $(j, \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}\,[\![\sigma]\!]\,\emptyset$, $(j, \mathsf{K}[\mathsf{v_1}], {}^{\sigma'}\mathcal{ST}\,\mathbf{K}[\mathcal{TS}^{\sigma}\,\mathsf{v_2}]) \in \mathcal{E}\,[\![\sigma']\!]\,\emptyset$.

By Lemma 10.4, there exists $\mathbf{v_2}$ such that $\mathcal{TS}^{\sigma}\,\mathsf{v_2} \longmapsto^{*} \mathbf{return\ v_2}$. Then by Lemma 7.11, it is sufficient to show that $(j, \mathsf{K}[\mathsf{v_1}], {}^{\sigma'}\mathcal{ST}\,\mathbf{K}[\mathbf{return\ v_2}]) \in \mathcal{E}\,[\![\sigma']\!]\,\emptyset$, which holds by hypothesis since $(j, \mathsf{v_1}, \mathbf{v_2}) \in \mathcal{V}^{+}[\![\sigma]\!]$. □

Lemma 10.12 (Cross Language Expression Relation closed under Anti Reduction)
If $(k, \mathsf{e}, {}^{\sigma}\mathcal{ST}\,\mathbf{e}) \in \mathrm{Atom}[\sigma]\emptyset$, $\mathsf{e} \longmapsto^{k_1} \mathsf{e}'$, $\mathbf{e} \longmapsto^{k_2} \mathbf{e}'$, $(k', \mathsf{e}', \mathbf{e}') \in \mathcal{E}^{\div}[\![\sigma]\!]$ *and* $k \le k' + \min(k_1, k_2)$ *then* $(k, \mathsf{e}, \mathbf{e}) \in \mathcal{E}^{\div}[\![\sigma]\!]$

Proof

Immediate by definition of the operational semantics and Lemma 7.11 □

Lemma 10.13 (Cross Language Value Relation Embeds in Expression Relation)
If $(k, \mathsf{v}, \mathbf{v}) \in \mathcal{V}^{+}[\![\sigma]\!]$ *then* $(k, \mathsf{v}, \mathbf{return_0\ v}) \in \mathcal{E}^{\div}[\![\sigma]\!]$

Proof

We need to show

$$(k, \mathsf{v}, \mathbf{return_0\ v}) \in \mathcal{E}^{\div}[\![\sigma]\!]$$

that is

$$(k, \mathsf{v}, {}^{\sigma}\mathcal{ST}\,\mathbf{return_0\ v}) \in \mathcal{E}\,[\![\sigma]\!]\,\emptyset$$

By definition of $\mathcal{V}^{+}[\![\sigma]\!]$, ${}^{\sigma}\mathcal{ST}\,\mathbf{return_0\ v} \longmapsto^{*} \mathsf{v}'$ such that $(k, \mathsf{v}, \mathsf{v}') \in \mathcal{V}\,[\![\sigma]\!]\,\emptyset$. Thus the result holds by Lemma 10.12 and Lemma 7.8. □

Theorem 10.14 (Translation preserves Semantics)

1. *If* $\Gamma \vdash \mathsf{v} : \sigma$*, and* $\Gamma \vdash \mathsf{v} : \sigma \rightsquigarrow_v \mathbf{v}$ *then* $\Gamma \vdash \mathsf{v} \approx_{+} \mathbf{v} : \sigma$.

2. *If* $\Gamma \vdash \mathsf{e} : \sigma$*, and* $\Gamma \vdash \mathsf{e} : \sigma \rightsquigarrow_e \mathbf{e}$ *then* $\Gamma \vdash \mathsf{e} \approx_{\div} \mathbf{e} : \sigma$.

Proof

We proceed by mutual induction on the structure of the translation judgments. We omit the cases for unit, sums, pairs, projections, and case. For each case, suppose $(k, \gamma, \boldsymbol{\gamma}) \in \mathcal{G}^{+}[\![\Gamma]\!]$.

1. Values:

Case $\mathsf{v} = \mathsf{x}, \Gamma \vdash \mathsf{x} : \sigma \rightsquigarrow_v \mathbf{x}$:

We need to show that there exists v' such that ${}^{\sigma}\mathcal{ST}\,\mathbf{return}\;\boldsymbol{\gamma}(\mathbf{v}) \longmapsto^{*} \mathsf{v}'$ such that for any $k \geq 0$, $(k, \gamma(\mathsf{x}), \mathsf{v}') \in \mathcal{V}[\![\sigma]\!]\,\emptyset$. This holds directly by definition of $\mathcal{G}^{+}[\![\Gamma]\!]$.

Case $\mathsf{v} = \lambda(\mathsf{x} : \sigma').\,\mathsf{e}$. Then $\sigma = \sigma' \to \sigma''$ and $\Gamma \vdash \mathsf{v} : \sigma' \to \sigma'' \rightsquigarrow_v \mathbf{v}$ where

$$
\begin{array}{l}
\mathbf{v} = \mathbf{pack}\,(\boldsymbol{\tau}_{\mathbf{env}}, \langle \lambda(\mathbf{z} : \langle \boldsymbol{\tau}_{\mathbf{env}}, \sigma'^{+} \rangle).\qquad\qquad , \langle \mathbf{y_1}, \ldots, \mathbf{y_n} \rangle \rangle)\;\mathbf{as}\;\exists \boldsymbol{\alpha}.\,\langle (\langle \boldsymbol{\alpha}, \sigma'^{+} \rangle \to \sigma''^{\div}), \boldsymbol{\alpha} \rangle, \\
\qquad\qquad\qquad \mathbf{let}\,\mathbf{y_{env}} = \mathbf{return_0}\;\mathbf{z.1}\;\mathbf{in} \\
\qquad\qquad\qquad \mathbf{let}\,\mathbf{y_1} = \mathbf{return_0}\;\mathbf{y_{env}.1}\;\mathbf{in} \\
\qquad\qquad\qquad \vdots \\
\qquad\qquad\qquad \mathbf{let}\,\mathbf{y_n} = \mathbf{return_0}\;\mathbf{y_{env}.n}\;\mathbf{in} \\
\qquad\qquad\qquad \mathbf{let}\,\mathbf{x} = \mathbf{return_0}\;\mathbf{z.2}\;\mathbf{in}\;\mathbf{e}
\end{array}
$$

$(\mathsf{y}_1, \ldots, \mathsf{y}_n) = \mathrm{fv}(\lambda(\mathsf{x} : \sigma').\,\mathsf{e})$, $\Gamma(\mathsf{y}_i) = \sigma_i$ for each $i \in \{1, \ldots, n\}$, $\Gamma' = (\mathsf{y}_1 : \sigma_1, \ldots, \mathsf{y}_n : \sigma_n)$, $\boldsymbol{\tau}_{\mathbf{env}} = \langle {\sigma_1}^{+}, \ldots, {\sigma_n}^{+} \rangle$, and $\Gamma', \mathsf{x} : \sigma \vdash \mathsf{e} : \sigma'' \rightsquigarrow_e \mathbf{e}$.
Next, ${}^{\sigma' \to \sigma''}\mathcal{ST}\,\boldsymbol{\gamma}(\mathbf{v}) \longmapsto \mathsf{v}_2$, where

$$
\begin{array}{l}
\mathsf{v}_2 = \lambda(\mathsf{x} : \sigma').\,{}^{\sigma''}\mathcal{ST}\,(\mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{z}') = \boldsymbol{\gamma}(\mathbf{v})\;\mathbf{in}\;\mathbf{let}\,\mathbf{x_f} = \mathbf{return}\;\mathbf{z'.1}\;\mathbf{in} \qquad\qquad) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathbf{let}\,\mathbf{x_{env}} = \mathbf{return}\;\mathbf{z'.2}\;\mathbf{in} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathbf{let}\,\mathbf{x}' = \mathcal{TS}^{\sigma'}\,\mathsf{x}\;\mathbf{in}\;\mathbf{x_f}\;[\boldsymbol{\alpha}]\;\langle \mathbf{x_{env}}, \mathbf{x}' \rangle
\end{array}
$$

We need to show that $(k, \gamma(\mathsf{v}), \mathsf{v}_2) \in \mathcal{V}[\![\sigma' \to \sigma'']\!]\,\emptyset$.
Let $j \leq k, (j, \mathsf{v}_1', \mathsf{v}_2') \in \mathcal{V}[\![\sigma']\!]$. We need to show that

$$
\begin{array}{l}
(j, \gamma(\mathsf{e})[\mathsf{v}_1'/\mathsf{x}], {}^{\sigma''}\mathcal{ST}\,(\mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{z}') = \boldsymbol{\gamma}(\mathbf{v})\;\mathbf{in}\;\mathbf{let}\,\mathbf{x_f} = \mathbf{return}\;\mathbf{z'.1}\;\mathbf{in} \qquad\qquad)) \in \mathcal{E}\,[\![\sigma'']\!]\,\emptyset \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbf{let}\,\mathbf{x_{env}} = \mathbf{return}\;\mathbf{z'.2}\;\mathbf{in} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbf{let}\,\mathbf{x}' = \mathcal{TS}^{\sigma'}\,\mathsf{v}_2'\;\mathbf{in}\;\mathbf{x_f}\;[\boldsymbol{\alpha}]\;\langle \mathbf{x_{env}}, \mathbf{x}' \rangle
\end{array}
$$

By Lemma 10.4, $\mathcal{TS}^{\sigma'}\,\mathsf{v}_2' \longmapsto^{*} \mathbf{return}\;\mathbf{v_2'}$ for some $\mathbf{v_2'}$. Then by Lemma 10.6, $(j, \mathsf{v}_1', \mathbf{v_2'}) \in \mathcal{V}^{+}[\![\sigma']\!]$.
Now define $\gamma'(\mathsf{y}_i) = \gamma(\mathsf{y}_i)$ for each $\mathsf{y}_i \in \mathrm{fv}(\mathsf{v})$ and $\gamma'(\mathsf{x}) = \mathsf{v}_1'$. Then $\gamma(\mathsf{e})[\mathsf{v}_1'/\mathsf{x}] = \gamma'(\mathsf{e})$ since $\mathsf{y}_1, \ldots, \mathsf{y}_n, \mathsf{x} = \mathrm{fv}(\mathsf{e})$.
Next, define $\boldsymbol{\gamma}'(\mathbf{y_i}) = \boldsymbol{\gamma}(\mathbf{y_i})$ for each $\mathsf{y}_i \in \mathrm{fv}(\mathsf{v})$ and $\boldsymbol{\gamma}'(\mathbf{x}) = \mathbf{v_2'}$. Then $(j, \gamma', \boldsymbol{\gamma}') \in \mathcal{G}^{+}[\![\Gamma', \mathsf{x} : \sigma']\!]$ by Lemma 7.6.
Next,

$$
\begin{array}{l}
{}^{\sigma''}\mathcal{ST}\,(\mathbf{unpack}\,(\boldsymbol{\alpha}, \mathbf{z}') = \boldsymbol{\gamma}(\mathbf{v})\;\mathbf{in}\;\mathbf{let}\,\mathbf{x_f} = \mathbf{return}\;\mathbf{z'.1}\;\mathbf{in} \qquad\qquad) \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathbf{let}\,\mathbf{x_{env}} = \mathbf{return}\;\mathbf{z'.2}\;\mathbf{in} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathbf{let}\,\mathbf{x}' = \mathcal{TS}^{\sigma'}\,\mathsf{v}_2'\;\mathbf{in}\;\mathbf{x_f}\;[\boldsymbol{\alpha}]\;\langle \mathbf{x_{env}}, \mathbf{x}' \rangle \\
\longmapsto^{*} \boldsymbol{\gamma}'(\mathbf{e}[\boldsymbol{\tau}/\boldsymbol{\alpha}][\ldots/\mathbf{z}'][\ldots/\mathbf{x_f}][\ldots/\mathbf{x_{env}}][\ldots/\mathbf{x}'][\ldots/\mathbf{z}][\ldots/\mathbf{y_{env}}]) \\
= \boldsymbol{\gamma}'(\mathbf{e})
\end{array}
$$

The last equality is justified by the fact that $\boldsymbol{\alpha}, \mathbf{z}', \mathbf{x_f}, \mathbf{x_{env}}, \mathbf{x}', \mathbf{z}, \mathbf{y_{env}} \notin \mathrm{fv}(\mathbf{e})$ which we know by Theorem 10.1.
Finally, by Lemma 7.11, we need to show that $(j, \gamma'(\mathsf{e}), \boldsymbol{\gamma}'(\mathbf{e}) \in \mathcal{E}^{\div}[\![\sigma'']\!]$ which holds by inductive hypothesis.

Case $\mathsf{v} = \mathsf{fold}_{\mu\alpha.\sigma'}\,\mathsf{v}'$: Then $\Gamma \vdash \mathsf{fold}_{\mu\alpha.\sigma}\,\mathsf{v} : \mu\alpha.\,\sigma \rightsquigarrow_v \mathbf{fold}_{\mu\alpha.\sigma^+}\,\mathbf{v}$ where $\Gamma \vdash \mathsf{v} : \sigma[\mu\alpha.\,\sigma/\alpha] \rightsquigarrow_v \mathbf{v}$.
By inductive hypothesis, $\exists \mathsf{v}_2.\ \gamma(\mathcal{TS}^{\,\sigma[\mu\alpha.\sigma/\alpha]}\,\mathbf{v}) \longmapsto^* \mathsf{v}_2 \wedge (k, \gamma(\mathsf{v}'), \mathsf{v}_2) \in \mathcal{V}\,[\![\sigma[\mu\alpha.\,\sigma/\alpha]]\!]\,\emptyset$.
Then by the operational semantics, $\mathcal{TS}^{\,\mu\alpha.\sigma}\,\mathbf{fold}_{\mu\alpha.\sigma^+}\,\mathbf{v} \longmapsto^* \mathsf{fold}_{\mu\alpha.\sigma'}\,\mathsf{v}_2$.
We need to show $(k, \mathsf{fold}_{\mu\alpha.\sigma'}\,\gamma(\mathsf{v}'), \mathsf{fold}_{\mu\alpha.\sigma'}\,\mathsf{v}_2) \in \mathcal{V}\,[\![\mu\alpha.\,\sigma']\!]\,\emptyset$.
If $k = 0$, this is trivial. Otherwise the result follows by Lemma 7.6.

2. Expressions:

Case $\mathsf{e} = \mathsf{v}, \Gamma \vdash \mathsf{v} : \sigma \rightsquigarrow_e \mathbf{return}\ \mathbf{v}$ where $\Gamma \vdash \mathsf{v} : \sigma \rightsquigarrow_v \mathbf{v}$: we need to show that

$$(k, \gamma(\mathsf{v}), {}^{\sigma}\mathcal{ST}\,\mathbf{return}\ \gamma(\mathbf{v})) \in \mathcal{E}\,[\![\sigma]\!]\,\emptyset.$$

By inductive hypothesis there is a v' such that ${}^{\sigma}\mathcal{ST}\,\gamma(\mathbf{v}) \longmapsto^* \mathsf{v}'$ and $(k, \gamma(\mathsf{v}), \mathsf{v}') \in \mathcal{V}\,[\![\sigma]\!]\,\emptyset$, so the result holds by Lemma 7.11 and Lemma 7.8.

Case $\mathsf{e} = \mathsf{v}_1\ \mathsf{v}_2$: Then

$$\begin{aligned}\Gamma \vdash \mathsf{v}_1\ \mathsf{v}_2 : \sigma_2 \rightsquigarrow_e\ & \mathbf{unpack}\,(\alpha, \mathbf{z}) = \mathbf{v_1}\ \mathbf{in}\\ & \mathbf{let}\,\mathbf{y_1} = \mathbf{return}\ \mathbf{z.1}\ \mathbf{in}\\ & \mathbf{let}\,\mathbf{y_2} = \mathbf{return}\ \mathbf{z.2}\ \mathbf{in}\\ & \mathbf{y_1}\ \langle \mathbf{y_2}, \mathbf{v_2}\rangle\end{aligned}$$

where $\Gamma \vdash \mathsf{v}_1 : \sigma_1 \to \sigma_2 \rightsquigarrow_v \mathbf{v_1}$ and $\Gamma \vdash \mathsf{v}_2 : \sigma_1 \rightsquigarrow_v \mathbf{v_2}$.
We need to show that

$$\left(\begin{aligned}k, \gamma(\mathsf{v}_1)\ \gamma(\mathsf{v}_2), & \mathbf{unpack}\,(\alpha, \mathbf{z}) = \gamma(\mathbf{v_1})\ \mathbf{in}\\ & \mathbf{let}\,\mathbf{y_1} = \mathbf{return}\ \mathbf{z.1}\ \mathbf{in}\\ & \mathbf{let}\,\mathbf{y_2} = \mathbf{return}\ \mathbf{z.2}\ \mathbf{in}\\ & \mathbf{y_1}\ \langle \mathbf{y_2}, \gamma(\mathbf{v_2})\rangle\end{aligned}\right) \in \mathcal{E}^{\div}[\![\sigma_2]\!]$$

By Lemma 10.10, it is sufficient to show that

$$\left(\begin{aligned}k, \gamma(\mathsf{v}_1)\ \gamma(\mathsf{v}_2), & \mathbf{unpack}\,(\alpha, \mathbf{z}) = \mathbf{v}_1'\ \mathbf{in}\\ & \mathbf{let}\,\mathbf{y_1} = \mathbf{return}\ \mathbf{z.1}\ \mathbf{in}\\ & \mathbf{let}\,\mathbf{y_2} = \mathbf{return}\ \mathbf{z.2}\ \mathbf{in}\\ & \mathbf{y_1}\ \langle \mathbf{y_2}, \gamma(\mathbf{v_2})\rangle\end{aligned}\right) \in \mathcal{E}^{\div}[\![\sigma_2]\!]$$

where $\mathcal{TS}^{\,\sigma_1\to\sigma_2}\ {}^{\sigma_1\to\sigma_2}\mathcal{ST}\,\mathbf{return}\ \gamma(\mathbf{v_1}) \longmapsto^* \mathbf{return}\ \mathbf{v}_1'$. By definition of the operational semantics we see that

$$\mathbf{v}_1' = \mathbf{pack}\,(\mathbf{1}, \langle \lambda(\mathbf{z} : \langle \mathbf{1}, \sigma''^{+}\rangle).\ \mathcal{TS}^{\,\sigma'}\left(\begin{aligned}&\mathsf{let}\,\mathsf{x} = {}^{\sigma''}\mathcal{ST}\,\mathbf{return_0}\ \mathbf{z.2}\ \mathsf{in}\\ &\mathsf{v}_1'\ \mathsf{x}\end{aligned}\right), \langle\rangle\rangle)\ \mathbf{as}\,(\sigma_1 \to \sigma_2)^+$$

where

$$\begin{aligned}\mathsf{v}_1' = \lambda(\mathsf{x} : \sigma_1).\ {}^{\sigma_2}\mathcal{ST}\,(\mathbf{unpack}\,(\alpha, \mathbf{z}) = \mathbf{v_1}\ \mathbf{in}\ & \mathbf{let}\,\mathbf{x_f} = \mathbf{return}\ \mathbf{z.1}\ \mathbf{in}\\ & \mathbf{let}\,\mathbf{x_{env}} = \mathbf{return}\ \mathbf{z.2}\ \mathbf{in}\\ & \mathbf{let}\,\mathbf{x} = \mathcal{TS}^{\,\sigma_1}\,\mathsf{x}\ \mathbf{in}\ \mathbf{x_f}\ [\alpha]\ \langle \mathbf{x_{env}}, \mathbf{x}\rangle\ )\end{aligned}$$

By definition or $\mathcal{V}^+[\![]\!]$ and inductive hypothesis, $(k, \gamma(\mathsf{v}_1), \mathsf{v}_1') \in \mathcal{V}[\![\sigma_1 \to \sigma_2]\!]\,\emptyset$.
Next,

$$\begin{aligned}
{}^{\sigma_2}\mathcal{ST}\,&\mathbf{unpack}\,(\alpha, \mathbf{z}) = \mathbf{v}_1' \mathbf{\ in} \longmapsto^5 {}^{\sigma_2}\mathcal{ST}\left(\left(\begin{array}{l} \boldsymbol{\lambda}(\mathbf{z} : \langle \mathbf{1}, \sigma_1{}^+\rangle). \\ \quad \mathcal{TS}^{\sigma_2}\left(\begin{array}{l}\mathsf{let\,x} = {}^{\sigma_1}\mathcal{ST}\,\mathbf{return_0\ z.2}\ \mathsf{in} \\ \mathsf{v}_1'\ \mathsf{x}\end{array}\right)\end{array}\right)\ \langle\langle\rangle, \gamma(\mathbf{v_2})\rangle\right) \\
&\mathbf{let\ y_1 = return\ z.1\ in} \\
&\mathbf{let\ y_2 = return\ z.2\ in} \\
&\mathbf{y_1}\ \langle \mathbf{y_2}, \gamma(\mathbf{v_2})\rangle \\
&\longmapsto^2 {}^{\sigma_2}\mathcal{ST}\,\mathcal{TS}^{\sigma_2}\,(\mathsf{let\,x} = {}^{\sigma_1}\mathcal{ST}\,\mathbf{return_0}\ \gamma(\mathbf{v_2})\ \mathsf{in\,v}_1'\ \mathsf{x})
\end{aligned}$$

Therefore by Lemma 7.11 and Lemma 10.6, it is sufficient to show that

$$(k, \gamma(\mathsf{v}_1)\ \gamma(\mathsf{v}_2), \mathsf{let\,x} = {}^{\sigma_1}\mathcal{ST}\,\mathbf{return_0}\ \gamma(\mathbf{v_2})\ \mathsf{in\,v}_1'\ \mathsf{x}) \in \mathcal{E}[\![\sigma_2]\!]\,\emptyset.$$

Next, by Lemma 10.5, ${}^{\sigma_1}\mathcal{ST}\,\mathbf{return_0}\ \gamma(\mathbf{v_2}) \longmapsto^* \mathsf{v}_2''$ and by inductive hypothesis $(k, \gamma(\mathsf{v}_2), \mathsf{v}_2'') \in \mathcal{V}[\![\sigma_1]\!]\,\emptyset$, so $\mathsf{let\,x} = {}^{\sigma_1}\mathcal{ST}\,\mathbf{return_0}\ \gamma(\mathbf{v_2})\ \mathsf{in\,v}_1'\ \mathsf{x} \longmapsto^* \mathsf{v}_1'\ \mathsf{v}_2''$, so by Lemma 7.11 it is sufficient to show that

$$(k, \gamma(\mathsf{v}_1)\ \gamma(\mathsf{v}_2), \mathsf{v}_1'\ \mathsf{v}_2'') \in \mathcal{E}[\![\sigma_2]\!]\,\emptyset,$$

which holds by similar reasoning to Lemma 7.19.

Case $\mathsf{e} = \mathsf{unfold\,v}, \Gamma \vdash \mathsf{e} : \sigma[\mu\alpha.\,\sigma/\alpha] \rightsquigarrow_e \mathbf{return\ unfold\ v}$, where $\Gamma \vdash \mathsf{v} : \mu\alpha.\,\sigma \rightsquigarrow_v \mathbf{v}$.
We need to show that for all $k \geq 0$,

$$(k, \mathsf{unfold}\,\gamma(\mathsf{v}), {}^{\sigma[\mu\alpha.\sigma/\alpha]}\mathcal{ST}\,\mathbf{return\ unfold}\,\gamma(\mathbf{v})) \in \mathcal{E}\left[\!\!\left[\sigma[\mu\alpha.\,\sigma/\alpha]^{\div}\right]\!\!\right]\emptyset$$

and by inductive hypothesis and Lemma 10.5, ${}^{\mu\alpha.\sigma}\mathcal{ST}\,\mathbf{return}\ \gamma(\mathbf{v}) \longmapsto^* \mathsf{v}'$ and $(k, \gamma(\mathsf{v}), \mathsf{v}') \in \mathcal{V}[\![\mu\alpha.\,\sigma]\!]\,\emptyset$. By definition of $\mathcal{V}[\![\mu\alpha.\,\sigma]\!]\,\emptyset$, this means $\gamma(\mathsf{v}) = \mathsf{fold}_{\mu\alpha.\sigma}\,\mathsf{v}_1$ and $\mathsf{v}' = \mathsf{fold}_{\mu\alpha.\sigma}\,\mathsf{v}_2$ where for all $j < k$, $(j, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\sigma[\mu\alpha.\,\sigma/\alpha]]\!]\,\emptyset$.
Then by definition of the operational semantics, $\gamma(\mathbf{v}) = \mathbf{fold}_{(\mu\alpha.\sigma)^+}\ \mathbf{v_2}$ where ${}^{\sigma[\mu\alpha.\sigma/\alpha]}\mathcal{ST}\,\mathbf{return\ v_2} \longmapsto^* \mathsf{v}_2$.
Therefore

$$\mathsf{unfold}\,\gamma(\mathsf{v}) = \mathsf{unfold\,fold}_{\mu\alpha.\sigma}\,\mathsf{v}_1 \longmapsto \mathsf{v}_1$$

and

$$\begin{aligned}
{}^{\sigma[\mu\alpha.\sigma/\alpha]}\mathcal{ST}\,\mathbf{return\ unfold}\,\gamma(\mathbf{v}) &= {}^{\sigma[\mu\alpha.\sigma/\alpha]}\mathcal{ST}\,\mathbf{return\ unfold}\,(\mathbf{fold}_{(\mu\alpha.\sigma)^+}\ \mathbf{v_2}) \\
&\longmapsto {}^{\sigma[\mu\alpha.\sigma/\alpha]}\mathcal{ST}\,\mathbf{return\ v_2} \\
&\longmapsto^* \mathsf{v}_2
\end{aligned}$$

so by Lemma 7.11, it is sufficient to show that $(k-1, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{E}[\![\sigma[\mu\alpha.\,\sigma/\alpha]]\!]\,\emptyset$, which holds by inductive hypothesis and Lemma 7.8.

Case $\mathsf{e} = \mathsf{let\,x{=}e_1\,in\,e_2}, \Gamma \vdash \mathsf{e} : \sigma \rightsquigarrow_e \mathbf{handle\ e_1\ with\ (x.\,e_2)\ (y.\,raise\ y)}$, where $\Gamma \vdash \mathsf{e}_1 : \sigma' \rightsquigarrow_e \mathbf{e_1}$ and $\Gamma, \mathsf{x} : \sigma' \vdash \mathsf{e}_2 : \sigma \rightsquigarrow_e \mathbf{e_2}$.
We need to show that for all $k \geq 0$,

$$(k, \mathsf{let\,x} = \gamma(\mathsf{e}_1)\ \mathsf{in}\ \gamma(\mathsf{e}_2), {}^{\sigma}\mathcal{ST}\,(\mathbf{handle}\ \gamma(\mathbf{e_1})\ \mathbf{with\ (x.}\,\gamma(\mathbf{e_2}))\ \mathbf{(y.\,raise\ y)})) \in \mathcal{E}\left[\!\!\left[\sigma^{\div}\right]\!\!\right]\emptyset.$$

By Lemma 10.10, it is sufficient to show that

$$(k, \mathsf{let\,x{=}}\gamma(\mathsf{e}_1)\ \mathsf{in}\ \gamma(\mathsf{e}_2), {}^{\sigma}\mathcal{ST}\,(\mathbf{handle}\ \mathcal{TS}^{\sigma'}\,{}^{\sigma'}\mathcal{ST}\,\gamma(\mathbf{e_1})\ \mathbf{with\ (x.}\,\gamma(\mathbf{e_2}))\ \mathbf{(y.\,raise\ y)})) \in \mathcal{E}\left[\!\!\left[\sigma^{\div}\right]\!\!\right]\emptyset)$$

By inductive hypothesis, $(k, \gamma(\mathsf{e}_1), {}^{\sigma'}\mathcal{ST}\,\boldsymbol{\gamma}(\mathbf{e_1})) \in \mathcal{E}\left[\!\left[\sigma'^{\div}\right]\!\right]\emptyset$. By Lemma 7.9, it is sufficient to show that for all $j \leq k, (j, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\sigma']\!]\,\emptyset$,

$$(j, \mathsf{let}\,\mathsf{x} = \mathsf{v}_1\,\mathsf{in}\,\gamma(\mathsf{e}_2), {}^{\sigma}\mathcal{ST}\,(\mathbf{handle}\,\mathcal{TS}^{\sigma'}\,\mathsf{v}_2\,\mathbf{with}\,(\mathbf{x}.\,\boldsymbol{\gamma}(\mathbf{e_2}))\,(\mathbf{y}.\,\mathbf{raise}\,\mathbf{y}))) \in \mathcal{E}\left[\!\left[\sigma'^{\div}\right]\!\right]\emptyset.$$

By Lemma 10.4, there exists $\mathbf{v_2}$ such that $\mathcal{TS}^{\sigma'}\,\mathsf{v}_2 \longmapsto^{*} \mathbf{return}\ \mathbf{v_2}$. Define $\gamma' = \gamma[\mathsf{x} \mapsto \mathsf{v}_1], \boldsymbol{\gamma'} = \boldsymbol{\gamma}[\mathbf{x} \mapsto \mathbf{v_2}]$. Then by Lemma 10.9, $(k, \gamma', \boldsymbol{\gamma'}) \in \mathcal{G}^{+}[\![\Gamma, \mathsf{x} : \sigma']\!]$.
Finally,

$$\mathsf{let}\,\mathsf{x} = \mathsf{v}_1\,\mathsf{in}\,\gamma(\mathsf{e}_2) \longmapsto \gamma'(\mathsf{e}_2)$$

and

$${}^{\sigma}\mathcal{ST}\,(\mathbf{handle}\,\mathcal{TS}^{\sigma'}\,\mathsf{v}_2\,\mathbf{with}\,(\mathbf{x}.\,\boldsymbol{\gamma}(\mathbf{e_2}))\,(\mathbf{y}.\,\mathbf{raise}\,\mathbf{y})) \longmapsto {}^{\sigma}\mathcal{ST}\,\boldsymbol{\gamma'}(\mathbf{e_2})$$

So by Lemma 7.11, it is sufficient to show that $(j, \gamma'(\mathsf{e}_2), {}^{\sigma}\mathcal{ST}\,\boldsymbol{\gamma'}(\mathbf{e_2})) \in \mathcal{E}[\![\sigma^{+}]\!]\,\emptyset$, which holds by inductive hypothesis.

□

Lemma 10.15 (Translation and Back-Translation Preserves and Reflects Termination)

1. *If* $\cdot \vdash \mathsf{e} : \sigma \rightsquigarrow_e \mathbf{e}$ *then* $\mathsf{e} \Downarrow$ iff $\mathbf{e} \Downarrow$.
2. *If* $\cdot; \cdot \vdash^{\div} \mathbf{e} : \boldsymbol{\theta} \twoheadrightarrow \mathsf{e}_u$ *then* $\mathbf{e} \Downarrow$ iff $\mathsf{e}_u \Downarrow$

Proof

By Lemma 10.14, $\cdot \vdash \mathsf{e} \approx_{\div} \mathbf{e} : \sigma$. Unfolding definitions, we get $\forall k, (k, \mathsf{e}, {}^{\sigma}\mathcal{ST}\,\mathbf{e}) \in \mathcal{E}[\![\sigma]\!]\,\emptyset$. Choosing $(k, [\cdot], [\cdot]) \in \mathcal{K}[\![\sigma]\!]\,\emptyset$, we get that $\forall k, (k, \mathsf{e}, {}^{\sigma}\mathcal{ST}\,\mathbf{e}) \in \mathcal{O}$.
Then if $\mathsf{e} \longmapsto^{j} \mathsf{v}$, since $(j+1, \mathsf{e}, {}^{\sigma}\mathcal{ST}\,\mathbf{e}) \in \mathcal{O}$, ${}^{\sigma}\mathcal{ST}\,\mathbf{e} \Downarrow$. Furthermore, if ${}^{\sigma}\mathcal{ST}\,\mathbf{e} \Downarrow$ then $\mathbf{e} \Downarrow$.
The other direction can be proved by a symmetric argument by starting with $\forall k, (k, \mathcal{TS}^{\sigma}\,\mathsf{e}, \mathbf{e}) \in \mathcal{E}[\![\sigma^{\div}]\!]\,\emptyset$.

By Theorem 9.11, $\cdot; \cdot \vdash \mathsf{e}_u \approx^{log}_{\mathcal{E}^{\mathsf{U}}} \mathbf{e} : \boldsymbol{\theta}$. Unfolding definitions we get $\forall k, (k, \mathsf{e}_u, \mathbf{e}) \in \mathcal{E}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\emptyset$.
Then we have $\forall k, (k, \mathsf{let}\,\mathsf{x}{=}[\cdot]\,\mathsf{in}\,\langle\rangle, \mathcal{TS}^{\langle\,\rangle}\,\mathbf{handle}\,[\cdot]\,\mathbf{with}\,(\mathbf{x}.\,\mathbf{return}\,\langle\,\rangle)\,(\mathbf{y}.\,\mathbf{return}\,\langle\,\rangle)) \in \mathcal{K}^{\mathsf{U}}[\![\boldsymbol{\theta}]\!]\emptyset$.
Then if $\mathsf{e}_u \longmapsto^{j} \mathsf{v}_u$, $(j+2, \mathsf{let}\,\mathsf{x} = \mathsf{e}_u\,\mathsf{in}\,\langle\rangle, \mathcal{TS}^{\langle\,\rangle}\,\mathbf{let}\,\mathbf{x} = \mathbf{e}\,\mathbf{in}\,\langle\,\rangle) \in \mathcal{O}$ and $\mathsf{let}\,\mathsf{x} = \mathsf{e}_u\,\mathsf{in}\,\langle\rangle \not\longmapsto^{j+2}$, so $\mathcal{TS}^{\langle\,\rangle}\,\mathbf{let}\,\mathbf{x} = \mathbf{e}\,\mathbf{in}\,\langle\,\rangle \Downarrow$, and therefore $\mathbf{e} \Downarrow$.
A similar argument gives the reverse implication.

□

10.2 Full Abstraction

Lemma 10.16 (Translation is Equivalent to Embedding)
If $\mathsf{e} \in \lambda^{\mathrm{S}}$ *and* $\Gamma \vdash \mathsf{e} : \sigma$ *and* $\Gamma \vdash \mathsf{e} : \sigma \rightsquigarrow_e \mathbf{e}$, *and* $\Gamma = \mathsf{x}_1 : \sigma_1, \ldots, \mathsf{x}_n : \sigma_n$ *then*

$$\begin{array}{l} \cdot; \Gamma^{+} \vdash \mathbf{e} \approx^{ctx}_{\mathrm{ST}} \mathcal{TS}^{\sigma}\,\mathsf{let}\,\mathsf{x}_1 = {}^{\sigma_1}\mathcal{ST}\,\mathbf{return}\ \mathbf{x_1}\ \mathsf{in}\ \ : \sigma^{\div}. \\ \qquad\qquad\qquad\qquad\quad \vdots \\ \qquad\qquad\qquad\qquad\quad \mathsf{let}\,\mathsf{x}_n = {}^{\sigma_n}\mathcal{ST}\,\mathbf{return}\ \mathbf{x_n}\ \mathsf{in} \\ \qquad\qquad\qquad\qquad\quad \mathsf{e} \end{array}$$

We denote the term on the right as $\mathcal{TS}^{\sigma}\,\mathsf{let}\,\Gamma = \ \mathcal{ST}\,\Gamma^{+}\,\mathsf{in}\,\mathsf{e}$.

Proof

By Theorem 7.43, it is sufficient to show that $\cdot;\Gamma^{+} \vdash \mathbf{e} \approx^{log}_{\mathcal{E}} \mathcal{TS}^{\,\sigma}\ \mathsf{let}\,\Gamma = \ \mathcal{ST}\,\Gamma^{+}\ \mathsf{in}\,\mathsf{e} : \sigma^{\div}$.

Suppose $(k, \gamma) \in \mathcal{G}\,[\![\Gamma^{+}]\!]\,\emptyset$. Then by Lemma 10.5, for each $\mathsf{x}_i : \sigma_i \in \Gamma$,

$$^{\sigma_i}\mathcal{ST}\,\mathbf{return}\ \gamma_2(\mathbf{x_i}) \longmapsto^{*} \mathsf{v}_i$$

for some v_i and $(k, \mathsf{v}_i, \gamma_2(\mathbf{x_i})) \in \mathcal{V}^{+}[\![\sigma_i]\!]$.

Therefore

$$\mathcal{TS}^{\,\sigma}\ \mathsf{let}\,\Gamma = \ \mathcal{ST}\,\gamma_2(\Gamma^{+})\ \mathsf{in}\,\gamma_2(\mathsf{e}) \longmapsto^{*} \mathcal{TS}^{\,\sigma}\,(\cdots\gamma_2(\mathsf{e})[\mathsf{v}_1/\mathsf{x}_1]\cdots)[\mathsf{v}_n/\mathsf{x}_n]$$

Next, $\gamma_2(\mathsf{x}_i) = \mathsf{v}_i$ since $\Gamma \cap \Gamma^{+} = \emptyset$. Define $\gamma(\mathsf{x}_i) = \mathsf{v}_i$ for each $\mathsf{x}_i \in \Gamma$.

Next we want to show that $(k, \gamma, \gamma_1) \in \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]$. For any $\mathsf{x}_i : \sigma_i$, we have $(k, \gamma(\mathsf{x}_i) = \mathsf{v}_i, \gamma_2(\mathbf{x_i})) \in \mathcal{V}^{+}[\![\sigma_i]\!]$ and $(k, \gamma_2(\mathbf{x_i}), \gamma_1(\mathbf{x_i})) \in \mathcal{V}\,[\![\sigma_i{}^{+}]\!]\,\emptyset$. But by Lemma 7.35, $^{\sigma_i}\mathcal{ST}\,\mathbf{return}\ \gamma_1(\mathbf{x_i}) \longmapsto^{*} \mathsf{v}'_i$ and $(k, \mathsf{v}_i, \mathsf{v}'_i) \in \mathcal{V}\,[\![\sigma_i]\!]\,\emptyset$, that is, $(k, \mathsf{v}_i, \gamma_1(\mathbf{x_i})) \in \mathcal{V}^{+}[\![\sigma_i]\!]$. Then we have $(k, \gamma, \gamma_1) \in \mathcal{G}^{\mathsf{U}}[\![\Gamma]\!]$.

Then by Lemma 7.11, it is sufficient to show that

$$(k, \mathcal{TS}^{\,\sigma}\,\gamma(\mathsf{e}), \boldsymbol{\gamma}(\mathbf{e})) \in \mathcal{E}\,[\![\sigma^{\div}]\!]\,\emptyset$$

which by Lemma 10.9 is equivalent to showing

$$(k, \gamma(\mathsf{e}), \boldsymbol{\gamma}(\mathbf{e})) \in \mathcal{E}^{\div}[\![\sigma]\!]$$

which follows from Lemma 10.14. □

Theorem 10.17 (Source Equivalence Implies Multi-language Equivalence)
If $\mathsf{e}_1, \mathsf{e}_2 \in \lambda^{\mathrm{S}}$ *and* $\Gamma \vdash \mathsf{e}_1 \approx^{ctx}_{\mathsf{S}} \mathsf{e}_2 : \sigma$*, then* $\cdot;\Gamma \vdash \mathsf{e}_1 \approx^{ctx}_{\mathrm{ST}} \mathsf{e}_2 : \sigma$.

Proof

We show one direction of the equivalence, the other follows by symmetry.

Suppose $C \in \lambda^{\mathrm{ST}}$ is an appropriate closing context and $C[\mathsf{e}_1]\Downarrow$. We need to show that $C[\mathsf{e}_2]\Downarrow$.

By Lemma 9.14 and Lemma 9.15, we back-translate $\cdot;\cdot \vdash C[\mathsf{e}_1] : \sigma' \twoheadrightarrow \mathsf{C}[\mathsf{e}_1]$ and $\cdot;\cdot \vdash C[\mathsf{e}_2] : \sigma' \twoheadrightarrow \mathsf{C}[\mathsf{e}_2]$ where $\mathsf{C} \in \lambda^{\mathrm{S}}$.

By Lemma 10.15, $C[\mathsf{e}_1]\Downarrow$ iff $\mathsf{C}[\mathsf{e}_1]\Downarrow$ and $C[\mathsf{e}_2]\Downarrow$ iff $\mathsf{C}[\mathsf{e}_2]\Downarrow$.

Since $\mathsf{C} \in \lambda^{\mathrm{S}}$ and $\Gamma \vdash \mathsf{e}_1 \approx^{ctx}_{\mathsf{S}} \mathsf{e}_2 : \sigma$, $\mathsf{C}[\mathsf{e}_1]\Downarrow$ iff $\mathsf{C}[\mathsf{e}_2]\Downarrow$.

Then we compose the iffs, to get the result:

$$\mathsf{C}[\mathsf{e}_1]\Downarrow \text{iff}\, C[\mathsf{e}_1]\Downarrow \text{iff}\, C[\mathsf{e}_2]\Downarrow \text{iff}\, \mathsf{C}[\mathsf{e}_2]\Downarrow .$$

□

Theorem 10.18 (Translation Preserves Multi-language Equivalence)
If $\cdot;\Gamma \vdash \mathsf{e}_1 \approx^{ctx}_{\mathrm{ST}} \mathsf{e}_2 : \sigma$, $\Gamma \vdash \mathsf{e}_1 : \sigma \rightsquigarrow_e \mathbf{e_1}$ *and* $\Gamma \vdash \mathsf{e}_2 : \sigma \rightsquigarrow_e \mathbf{e_2}$*, then* $\cdot;\Gamma^{+} \vdash \mathbf{e_1} \approx^{ctx}_{\mathrm{ST}} \mathbf{e_2} : \sigma^{\div}$.

Proof

By Lemma 10.16,

$$\cdot;\Gamma^{+} \vdash \mathbf{e} \approx^{ctx}_{\mathrm{ST}} \mathcal{TS}^{\,\sigma}\ \mathsf{let}\,\Gamma = \ \mathcal{ST}\,\Gamma^{+}\ \mathsf{in}\,\mathsf{e} : \sigma^{\div}$$

and

$$\cdot;\Gamma^{+} \vdash \mathbf{e}' \approx^{ctx}_{\mathrm{ST}} \mathcal{TS}^{\,\sigma}\ \mathsf{let}\,\Gamma = \ \mathcal{ST}\,\Gamma^{+}\ \mathsf{in}\,\mathsf{e}' : \sigma^{\div}.$$

Since $\cdot;\Gamma \vdash \mathsf{e}_1 \approx^{ctx}_{\mathrm{ST}} \mathsf{e}_2 : \sigma$,

$$\cdot;\Gamma^{+} \vdash \mathcal{TS}^{\,\sigma}\ \mathsf{let}\,\Gamma = \ \mathcal{ST}\,\Gamma^{+}\ \mathsf{in}\,\mathsf{e} \approx^{ctx}_{\mathrm{ST}} \mathcal{TS}^{\,\sigma}\ \mathsf{let}\,\Gamma = \ \mathcal{ST}\,\Gamma^{+}\ \mathsf{in}\,\mathsf{e}' : \sigma^{\div}.$$

The result then holds by transitivity of contextual equivalence. □

Theorem 10.19 (Multi-language Equivalence Implies Target Equivalence)
If $\cdot;\Gamma^+ \vdash \mathbf{e_1} \approx^{ctx}_{\mathrm{ST}} \mathbf{e_2} : \sigma^{\div}$*, then* $\cdot;\Gamma^+ \vdash \mathbf{e_1} \approx^{ctx}_{\mathbf{T}} \mathbf{e_2} : \sigma^{\div}$*.*

Proof

Trivial, since every target context is a multi-language context. □

Theorem 10.20 (Translation is Equivalence Preserving)
If $\Gamma \vdash \mathsf{e} \approx^{ctx}_{\mathsf{S}} \mathsf{e}' : \sigma$*,* $\Gamma \vdash \mathsf{e} : \sigma \rightsquigarrow_e \mathbf{e}$ *and* $\Gamma \vdash \mathsf{e}' : \sigma \rightsquigarrow_e \mathbf{e}'$ *then* $\cdot;\Gamma^+ \vdash \mathbf{e} \approx^{ctx}_{\mathbf{T}} \mathbf{e}' : \sigma^{\div}$*.*

Proof (1: Decomposed)

By composition of Theorem 10.17, Theorem 10.18 and Theorem 10.19. □

Proof (2: Direct)

We prove one direction, the other case holds by symmetry. Suppose $\mathbf{C} \in \lambda^{\mathrm{T}}$ appropriately typed.
By Lemma 10.16,

$$\cdot;\Gamma^+ \vdash \mathbf{e} \approx^{ctx}_{\mathrm{ST}} \mathcal{TS}^{\,\sigma}\ \mathsf{let}\ \Gamma = \ \mathcal{ST}\,\Gamma^+\ \mathsf{in}\,\mathsf{e} : \sigma^{\div}$$

and

$$\cdot;\Gamma^+ \vdash \mathbf{e}' \approx^{ctx}_{\mathrm{ST}} \mathcal{TS}^{\,\sigma}\ \mathsf{let}\ \Gamma = \ \mathcal{ST}\,\Gamma^+\ \mathsf{in}\,\mathsf{e}' : \sigma^{\div}.$$

Let $C = \mathbf{C}[\mathcal{TS}^{\,\sigma}\ \mathsf{let}\ \Gamma = \ \mathcal{ST}\,\Gamma^+\ \mathsf{in}\ [\cdot]]$. Then $\mathbf{C}[\mathsf{e}] \Downarrow$ iff $C[\mathsf{e}] \Downarrow$ and $\mathbf{C}[\mathsf{e}'] \Downarrow$ iff $C[\mathsf{e}'] \Downarrow$.
Next, by Lemma 9.15 and Lemma 9.14, we back-translate,

$$\cdot;\cdot \vdash C[\mathsf{e}] : \boldsymbol{\theta} \twoheadrightarrow \ \mathsf{C}[\mathsf{e}]$$

and

$$\cdot;\cdot \vdash C[\mathsf{e}'] : \boldsymbol{\theta} \twoheadrightarrow \ \mathsf{C}[\mathsf{e}'].$$

Then by Lemma 10.15, $C[\mathsf{e}] \Downarrow$ iff $\mathsf{C}[\mathsf{e}] \Downarrow$ and $C[\mathsf{e}'] \Downarrow$ iff $\mathsf{C}[\mathsf{e}'] \Downarrow$.
Then since $\Gamma \vdash \mathsf{e} \approx^{ctx}_{\mathsf{S}} \mathsf{e}' : \sigma$, $\mathsf{C}[\mathsf{e}] \Downarrow$ iff $\mathsf{C}[\mathsf{e}'] \Downarrow$.
Then we can compose the above iffs to get the result. In summary:

$$\mathbf{C}[\mathsf{e}] \Downarrow \text{iff}\, C[\mathsf{e}] \Downarrow \text{iff}\, \mathsf{C}[\mathsf{e}] \Downarrow \text{iff}\, \mathsf{C}[\mathsf{e}'] \Downarrow \text{iff}\, C[\mathsf{e}'] \Downarrow \text{iff}\, \mathbf{C}[\mathsf{e}'] \Downarrow .$$

□

Theorem 10.21 (Translation is Equivalence Reflecting)
If $\Gamma \vdash \mathsf{e} : \sigma \rightsquigarrow_e \mathbf{e}$*,* $\Gamma \vdash \mathsf{e}' : \sigma \rightsquigarrow_e \mathbf{e}'$ *and* $\cdot;\Gamma^+ \vdash \mathbf{e} \approx^{ctx}_{\mathbf{T}} \mathbf{e}' : \sigma^{\div}$ *then* $\Gamma \vdash \mathsf{e} \approx^{ctx}_{\mathsf{S}} \mathsf{e}' : \sigma$*.*

Proof

Assume $\vdash \mathsf{C} : (\cdot;\Gamma \vdash \sigma) \Rightarrow (\cdot;\cdot \vdash \sigma')$ We need to show that $\mathsf{C}[\mathsf{e}] \Downarrow$ iff $\mathsf{C}[\mathsf{e}] \Downarrow$.
First by Lemma 10.3, $\cdot \vdash \mathsf{C}[\mathsf{e}] : \sigma' \rightsquigarrow_e \mathbf{C}[\mathbf{e}]$ and $\cdot \vdash \mathsf{C}[\mathsf{e}'] : \sigma' \rightsquigarrow_e \mathbf{C}[\mathbf{e}']$.
Then by Lemma 10.15, $\mathbf{C}[\mathbf{e}] \Downarrow$ iff $\mathsf{C}[\mathsf{e}] \Downarrow$ and $\mathbf{C}[\mathbf{e}'] \Downarrow$ iff $\mathsf{C}[\mathsf{e}'] \Downarrow$.
Then since $\cdot;\Gamma^+ \vdash \mathbf{e} \approx^{ctx}_{\mathbf{T}} \mathbf{e}' : \sigma^{\div}$ and $\mathbf{C} \in \lambda^{\mathrm{T}}$, $\mathbf{C}[\mathbf{e}] \Downarrow$ iff $\mathbf{C}[\mathbf{e}'] \Downarrow$.
Finally, we compose the iffs to obtain our result:

$$\mathsf{C}[\mathsf{e}] \Downarrow \text{iff}\, \mathbf{C}[\mathbf{e}] \Downarrow \text{iff}\, \mathbf{C}[\mathbf{e}'] \Downarrow \text{iff}\, \mathsf{C}[\mathsf{e}'] \Downarrow$$

□

Theorem 10.22 (Translation is Fully Abstract)
If $\Gamma \vdash \mathsf{e} : \sigma \rightsquigarrow_e \mathbf{e}$ *and* $\Gamma \vdash \mathsf{e}' : \sigma \rightsquigarrow_e \mathbf{e}'$ *then* $\Gamma \vdash \mathsf{e} \approx^{ctx}_{\mathsf{S}} \mathsf{e}' : \sigma$ *if and only if* $\cdot;\Gamma^+ \vdash \mathbf{e} \approx^{ctx}_{\mathbf{T}} \mathbf{e}' : \sigma^{\div}$*.*

Proof

Immediate by Theorem 10.20 and Theorem 10.21 □