Simple Type Theory, Full Rules

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In simple type theory STT we have three fundamental notions: types A, terms $\Gamma \vdash M : A$ and equalities between terms $\Gamma \vdash M = N : A$.

As with IPL, the different connectives of STT $(1, \times, 0, +, \Rightarrow)$ are all presented independently, and so the system makes sense for any choice of which connectives to include. If we are speaking of a subsystem we will write the connectives explicitly, for instance STT$(1, \times, \Rightarrow)$ is simple type theory with only the singleton, product and function types. If we write STT by itself, it should mean the full system STT$(1, \times, 0, +, \Rightarrow)$.

In IPL in addition to the connectives we also considered propositional variables and axioms, which we grouped together as a *signature* $\Sigma = (\Sigma_0, \Sigma_1)$.

In STT, we have an analogous notion of signature which includes three things: base types, function symbols, and equational axioms. The base types are the analog of propositional variables, function symbols are the analog of IPL axioms, and equational axioms are fundamentally new.

Definition 1. *Given a set Σ_0 of base types, an STT type is one inductively generated by the base types in Σ_0, $0, 1$ and closed under $\times, +, \Rightarrow$. We call this set $STT(\Sigma_0)_{ty}$.*

An analogous definition can be provided for any subsystem of STT. For instance $STT(1, \times, \Rightarrow)(\Sigma_0)_{ty}$ is the set inductively generated by the base types in Σ_0, 1 and closed under $\times, \Rightarrow$.

Definition 2. *Given a set Σ_0 of base types, an* arity *is a pair of a finite sequence of STT types $STT(\Sigma_0)^*_{ty}$ and an output type $STT(\Sigma_0)_{ty}$. A function symbol is a name f and an arity. We write this as $f : A_0, \ldots \to B$.*

A set of function symbols relative to Σ_0 is a set Σ_1 of function symbols all of whose names are different.

Given a set of base types Σ_0 and a set Σ_1 of function symbols relative to Σ_1 we can define the terms $\Gamma \vdash_{STT(\Sigma_0,\Sigma_1)} M : A$ to be generated by the rules of simple type theory and the function symbol application rule:

$$\frac{f : A_0, \ldots \to B \in \Sigma_1 \qquad \Gamma \vdash_{STT(\Sigma_0,\Sigma_1)} M_0 : A_0 \qquad \cdots}{\Gamma \vdash_{STT(\Sigma_0,\Sigma_1)} f(M_0, \ldots) : B}$$

If Σ_0, Σ_1 are clear from the context then we will simply write $\Gamma \vdash M : A$.

Definition 3. *Given a set* Σ_0 *of function symbols and a set* Σ_1 *of axioms relative to it, an* equational axiom *is a quadrule* (Γ, A, M, N) *where* Γ, A *are well-formed relative to* Σ_0 *and* $\Gamma \vdash M : A$ *and* $\Gamma \vdash N : A$ *are well-formed relative to* Σ_0, Σ_1.

Definition 4. *An STT signature* Σ *is a triple* $(\Sigma_0, \Sigma_1, \Sigma_2)$ *of a set* Σ_0 *of base types, a set* Σ_1 *of function symbols relative to* Σ_0 *and a set* Σ_2 *of equational axioms relative to* Σ_0, Σ_1.

We write $STT(\Sigma)$ *for the system of types, terms and equations generated by simple type theory under* Σ.

Now fix a signature $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$.

Types (written A, B, C) are inductively generated as follows

$$\frac{X \in \Sigma_0}{X \text{ type}} \qquad \frac{}{1 \text{ type}} \qquad \frac{A \text{ type} \qquad B \text{ type}}{A \times B \text{ type}} \qquad \frac{A \text{ type} \qquad B \text{ type}}{A \Rightarrow B \text{ type}} \qquad \frac{}{0 \text{ type}}$$

$$\frac{A \text{ type} \qquad B \text{ type}}{A + B \text{ type}}$$

Contexts (written Γ, Δ, Ξ) are inductively generated as follows:

$$\cdot \text{ type} \qquad \frac{\Gamma \text{ type} \qquad A \text{ type}}{\Gamma, x : A \text{ type}}$$

where the variable x is assumed never to occur in Γ.

Terms are inductively generated as follows:

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}\ \textsc{Assumption} \qquad \frac{f : A_0, \ldots \to B \in \Sigma_1 \qquad \Gamma \vdash M_0 : A_0 \qquad \cdots}{\Gamma \vdash f(M_0, \ldots) : B}\ \textsc{FunSymbol}$$

$$\frac{}{\Gamma \vdash () : 1}\ 1\text{I} \qquad \frac{\Gamma \vdash M : 0}{\Gamma \vdash \text{case}_0\, M\{\} : B}\ 0\text{E}$$

$$\frac{\Gamma \vdash M_1 : A_1 \qquad \Gamma \vdash M_2 : A_2}{\Gamma \vdash (M_1, M_2) : A_1 \times A_2}\ \times\text{I} \qquad \frac{\Gamma \vdash N : A_1 \times A_2}{\Gamma \vdash \pi_1 N : A_1}\ \times\text{E1} \qquad \frac{\Gamma \vdash N : A_1 \times A_2}{\Gamma \vdash \pi_2 N : A_2}\ \times\text{E2}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}\ \Rightarrow\text{I} \qquad \frac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\, N : B}\ \Rightarrow\text{E}$$

$$\frac{\Gamma \vdash M_1 : A_1}{\Gamma \vdash i_1 M_1 : A_1 + A_2}\ +\text{I1} \qquad \frac{\Gamma \vdash M_2 : A_2}{\Gamma \vdash i_2 M_2 : A_1 + A_2}\ +\text{I2}$$

$$\frac{\Gamma \vdash M : A_1 + A_2 \qquad \Gamma, x_1 : A_1 \vdash N_1 : B \qquad \Gamma, x_1 : A_1 \vdash N_2 : B}{\Gamma \vdash \text{case}_+\, M\{i_1 x_1 \to N_1 | i_2 x_2 \to N_2\} : B}\ +\text{E}$$

Where in the λ rule, x is assumed not to occur in Γ and in the +E rule, x_1, x_2 are similarly assumed not to occur in Γ.

A substitution $\gamma : \Delta \to \Gamma$ is a function that takes every $x : A \in \Gamma$ to a well-typed term $\Delta \vdash \gamma(x) : A$. This can equivalently be described inductively as follows:

$$\frac{}{\cdot : \Delta \to \cdot} \qquad\qquad \frac{\gamma : \Delta \to \Gamma \qquad \Delta \vdash M : A}{\gamma, M/x : \Delta \to \Gamma, x : A}$$

And finally the equational theory of STT. The rules come in three groups. First the logical rules: reflexivity, transitivity, symmetry and the axioms.

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash M = M : A}\ \textsc{Refl} \qquad\qquad \frac{\Gamma \vdash N = M : A}{\Gamma \vdash M = N : A}\ \textsc{Sym}$$

$$\frac{\Gamma \vdash M = N : A \qquad \Gamma \vdash N = P : A}{\Gamma \vdash M = P : A}\ \textsc{Trans} \qquad\qquad \frac{(\Gamma, A, M, N) \in \Sigma_2}{\Gamma \vdash M = N : A}\ \textsc{Ax}$$

Then the congruence rules for each term former and substitution:

$$\frac{\Gamma, x : A \vdash M = M' : B \qquad \Gamma \vdash N = N : A}{\Gamma \vdash M[N/x] = M'[N'/x] : B}\ \textsc{SubstCong}$$

$$\frac{f : A_0, \ldots \to B \in \Sigma_1 \qquad \Gamma \vdash M_0 = M_0' : A_0 \qquad \cdots}{\Gamma \vdash f(M_0, \ldots) = f(M_0', \ldots) : B}\ \textsc{FunSymbCong}$$

$$\frac{\Gamma \vdash M = M' : 0}{\Gamma \vdash \mathrm{case}_0\, M\{\} = \mathrm{case}_0\, M'\{\} : B}\ 0\mathrm{E}\textsc{Cong}$$

$$\frac{\Gamma \vdash M_1 = M_1' : A_1 \qquad \Gamma \vdash M_2 = M_2' : A_2}{\Gamma \vdash (M_1, M_2) = (M_1', M_2') : A_1 \times A_2}\ \times\mathrm{I}\textsc{Cong}$$

$$\frac{\Gamma \vdash N = N' : A_1 \times A_2}{\Gamma \vdash \pi_1 N = \pi_1 N' : A_1}\ \times\mathrm{E1}\textsc{Cong} \qquad \frac{\Gamma \vdash N = N' : A_1 \times A_2}{\Gamma \vdash \pi_2 N = \pi_2 N' : A_2}\ \times\mathrm{E2}\textsc{Cong}$$

$$\frac{\Gamma, x : A \vdash M = M' : B}{\Gamma \vdash \lambda x.M = \lambda x.M' : A \Rightarrow B}\ \Rightarrow\mathrm{I}\textsc{Cong}$$

$$\frac{\Gamma \vdash M = M' : A \Rightarrow B \qquad \Gamma \vdash N = N' : A}{\Gamma \vdash M\, N = M'\, N' : B}\ \Rightarrow\mathrm{E}\textsc{Cong}$$

$$\frac{\Gamma \vdash M_1 = M_1' : A_1}{\Gamma \vdash i_1 M_1 = i_1 M_1' : A_1 + A_2}\ +\mathrm{I1}\textsc{Cong} \qquad \frac{\Gamma \vdash M_2 = M_2' : A_2}{\Gamma \vdash i_2 M_2 = i_2 M_2' : A_1 + A_2}\ +\mathrm{I2}\textsc{Cong}$$

$$\frac{\Gamma \vdash M = M' : A_1 + A_2 \qquad \Gamma, x_1 : A_1 \vdash N_1 = N_1' : B \qquad \Gamma, x_1 : A_1 \vdash N_2 = N_2' : B}{\Gamma \vdash \mathrm{case}_+\, M\{i_1 x_1 \to N_1 | i_2 x_2 \to N_2\} = \mathrm{case}_+\, M'\{i_1 x_1' \to N_1' | i_2 x_2' \to N_2'\} : B}\ +\mathrm{E}\textsc{Cong}$$

Finally the $\beta\eta$ rules.

$$\frac{\Gamma \vdash M : 1}{\Gamma \vdash M = () : 1}\ 1\eta \qquad \frac{\Gamma \vdash M_1 : A_1 \qquad \Gamma \vdash M_2 : A_2}{\Gamma \vdash \pi_1(M_1, M_2) = M_1 : A_1}\ \times\beta1$$

$$\frac{\Gamma \vdash M_1 : A_1 \qquad \Gamma \vdash M_2 : A_2}{\Gamma \vdash \pi_2(M_1, M_2) = M_2 : A_2}\ \times\beta2 \qquad \frac{\Gamma \vdash M : A_1 \times A_2}{\Gamma \vdash M = (\pi_1 M, \pi_2 M) : A_1 \times A_2}\ \times\eta$$

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x.M)\, N = M[N/x] : B} \Rightarrow \beta \qquad \frac{\Gamma \vdash M : A \Rightarrow B}{\Gamma \vdash M = \lambda x.M\, x : A \Rightarrow B} \Rightarrow \eta$$

$$\frac{x : 0 \in \Gamma \qquad \Gamma \vdash M : B}{\Gamma \vdash M = \mathrm{case}_0\, x\{\} : B}\ 0\eta$$

$$\frac{\Gamma \vdash M_1 : A_1 \qquad \Gamma, x_1 : A_1 \vdash N_1 : B \qquad \Gamma, x_2 : A_2 \vdash N_2 : B}{\Gamma \vdash \mathrm{case}_+\, i_1 M_1\{i_1 x_1 \to N_1 | i_2 x_2 \to N_2\} = N_1[M_1/x_1] : B}\ +\beta1$$

$$\frac{\Gamma \vdash M_2 : A_2 \qquad \Gamma, x_1 : A_1 \vdash N_1 : B \qquad \Gamma, x_2 : A_2 \vdash N_2 : B}{\Gamma \vdash \mathrm{case}_+\, i_2 M_2\{i_1 x_1 \to N_1 | i_2 x_2 \to N_2\} = N_1[M_1/x_1] : B}\ +\beta1$$

$$\frac{x : A_1 + A_2 \in \Gamma \qquad \Gamma \vdash M : B}{\Gamma \vdash M = \mathrm{case}_+\, x\{i_1 x_1 \to M[i_1 x_1/x] | i_2 x_2 \to M[i_2 x_2/x]\} : B}\ +\eta$$

The following equational reasoning principles are then admissible:

$$\frac{\Gamma \vdash M : 1 \qquad \Gamma \vdash N : 1}{\Gamma \vdash M = N : 1}\ 1\eta\textsc{Alt}$$

$$\frac{\begin{array}{c}\Gamma \vdash M : A_1 \times A_2 \qquad \Gamma \vdash N : A_1 \times A_2 \\ \Gamma \vdash \pi_1 M = \pi_1 N : A_1 \qquad \Gamma \vdash \pi_2 M = \pi_2 N : A_2\end{array}}{\Gamma \vdash M = N : A_1 \times A_2}\ \times\eta\textsc{Alt}$$

$$\frac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A \Rightarrow B \qquad \Gamma, x : A \vdash M\, x = N\, x : B}{\Gamma \vdash M = N : A \Rightarrow B} \Rightarrow \eta\textsc{Alt}$$

$$\frac{\Gamma \vdash P : 0 \qquad \Gamma \vdash M : B \qquad \Gamma \vdash N : B}{\Gamma \vdash M = N : B}\ 0\eta\textsc{Alt}$$

$$\frac{\begin{array}{c}\Gamma \vdash P : A_1 + A_2 \qquad \Gamma, x : A_1 + A_2 \vdash M : B \qquad \Gamma, x : A_1 + A_2 \vdash N : B \\ \Gamma, x_1 : A_1 \vdash M[i_1 x_1/x] = N[i_1 x_1/x] : B \qquad \Gamma, x_2 : A_2 \vdash M[i_2 x_2/x] = N[i_2 x_2/x] : B\end{array}}{\Gamma \vdash M[P/x] = N[P/x] : B}\ +\eta\textsc{Alt}$$