Problem Set 5

Released: March 6, 2023
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Modifications:

- Add several more definitions and the weak initiality theorems as a reference.
- Fix some notation Un_γ to match that used in class.
- Add assumption that $\mathcal{C}$ has a terminal object to problem 1.

Submit your solutions to this homework on Canvas in a group of 2 or 3. Your solutions must be submitted in pdf produced using LaTeX.

Definition 1. *Let $\mathcal{C}$ be a category*

- *An initial object in $\mathcal{C}$ is an object $0 \in \mathcal{C}$ such that for any $a \in \mathcal{C}$, there is a unique morphism*

$$[] : \mathcal{C}(0, a)$$

- *A binary coproduct structure for $a_1, a_2 \in \mathcal{C}$ consists of*
  - *An object $a_1 + a_2 \in \mathcal{C}$*
  - *Morphisms $i_1 : \mathcal{C}(a_1, a_1 + a_2)$ and $i_2 : \mathcal{C}(a_2, a_1 + a_2)$*
  - *Such that for every $g_1 : \mathcal{C}(a_1, b)$ and $g_2 : \mathcal{C}(a_2, b)$ there exists a unique $[g_1, g_2] : \mathcal{C}(a_1 + a_2, b)$ satisfying $[g_1, g_2] \circ i_1 = g_1$ and $[g_1, g_2] \circ i_2 = g_2$.*

Definition 2. *Let $\mathcal{C}$ be a category with binary products.*

An initial object $0 \in \mathcal{C}$ is distributive *if for every $a \in \mathcal{C}$ the unique morphism*

$$0 \to a \times 0$$

is an isomorphism.

A binary coproduct $a_1 + a_2$ with injections $i_1 : a_1 \to a_1 + a_2$ and $i_2 : a_2 \to a_1 + a_2$ is distributive *if for every $b \in \mathcal{C}$, the morphism*

$$[id_b \times i_1, id_b \times i_2] : (b \times a_1) + (b \times a_2) \to b \times (a_1 + a_2)$$

is an isomorphism.

Definition 3. *A CT structure $\mathcal{S}$ consists of*

1. *A category $\mathcal{S}_c$*
2. *A set $\mathcal{S}_T$.*
3. *For each type $A \in S_T$ a predicator $\mathrm{Tm}(A)$ on $\mathcal{S}_c$.*
4. *A terminal object $1 \in \mathcal{S}_c$*
5. *For each $\Gamma_1, \Gamma_2 \in \mathcal{S}_c$ a product structure $(\Gamma_1 \times \Gamma_2, \pi_1, \pi_2)$ for Γ_1, Γ_2, that is*
   - *An object $\Gamma_1 \times \Gamma_2 \in \mathcal{S}_c$*
   - *Morphisms $\pi_1^{\Gamma_1,\Gamma_2} : \Gamma_1 \times \Gamma_2 \to \Gamma_1$ and $\pi_2^{\Gamma_1,\Gamma_2} : \Gamma_1 \times \Gamma_2 \to \Gamma_2$.*
   - *Such that for any $\Delta \in \mathcal{S}_c$ and $f_1 : \Delta \to \Gamma_1$ and $f_2 : \Delta \to \Gamma_2$ there exists a unique $(f_1, f_2) : \Delta \to \Gamma_1 \times \Gamma_2$ such that $\pi_1^{\Gamma_1,\Gamma_2} \circ (f_1, f_2) = f_1$ and $\pi_2^{\Gamma_1,\Gamma_2} \circ (f_1, f_2) = f_2$.*
6. *For each $A \in \mathcal{S}_T$ a singleton context structure $(\mathrm{sole}A, \mathrm{var})$ for A, that is,*
   - *An object $\mathrm{sole}A \in \mathcal{S}_c$*
   - *An element $\mathrm{var}^A \in \mathrm{Tm}(A)(\mathrm{sole}A)$*
   - *Such that for any $\Gamma \in \mathcal{S}_c$ and $M \in \mathrm{Tm}(A)(\Gamma)$, there exists a unique $M/\mathrm{var}^A \in \Gamma \to \mathrm{sole}A$ such that $\mathrm{var}^A * M/\mathrm{var}^A = M$.*

Definition 4. *Let $\mathcal{S}$ be a CT structure and $\Gamma \in \mathcal{S}_c$. Define a category Un_Γ as follows:*

- $(\mathrm{Un}_\Gamma)_0 = \mathcal{S}_T$
- $(\mathrm{Un}_\Gamma)_1(A, B) = \mathrm{Tm}_\mathcal{S} B(\Gamma \times \mathrm{sole}A)$
- *With identity*
$$id_A = \mathrm{var}^A * (\pi_2^{\Gamma,\mathrm{sole}A})$$
- *composition of $M \in \mathrm{Tm}_\mathcal{S} C(\Gamma \times \mathrm{sole}B)$ and $N \in \mathrm{Tm}_\mathcal{S} B(\Gamma \times \mathrm{sole}A)$ defined as*
$$M \circ N = M * (\pi_1^{\Gamma,\mathrm{sole}A}, N/\mathrm{var}^B)$$
- *Identity and associativity properties follow from properties of products and the singleton contexts.*

Let $\gamma \in \mathcal{S}_c(\Delta, \Gamma)$, then we define a functor $\mathrm{Un}_\gamma : \mathrm{Un}_\Gamma \to \mathrm{Un}_\Delta$ as

$$\mathrm{Un}_\gamma(A) = A$$
$$\mathrm{Un}_\gamma(M) = M * (\gamma \circ \pi_1, \pi_2)$$

This preserves identity and composition again by properties of products and singleton contexts.

Definition 5. *Let $\mathcal{S}$ be a CT structure.*

- *A unit type in $\mathcal{S}$ is a type $1 \in \mathcal{S}_T$ such that for every $\Gamma \in \mathcal{S}_c$ there exists a unique term $() \in \mathrm{Tm}(1)(\Gamma)$.*
- *A product of types $A_1, A_2 \in \mathcal{S}_T$ is a type $A_1 \times A_2 \in \mathcal{S}_T$ with terms $\pi_1 \in \mathrm{Tm}(A_1)(\mathrm{sole}(A_1 \times A_2))$ and $\pi_2 \in \mathrm{Tm}(A_2)(\mathrm{sole}(A_1 \times A_2))$ such that for any pair of terms $M_1 \in \mathrm{Tm}(A_1)(\Gamma)$ and $M_2 \in \mathrm{Tm}(A_2)(\Gamma)$ there exists a unique term $(M_1, M_2) \in \mathrm{Tm}(A_1 \times A_2)(\Gamma)$ satisfying $\pi_1 * (M_1, M_2) = M_1$ and $\pi_2 * (M_1, M_2) = M_2$.*
- *An exponential of types $A, B \in \mathcal{S}_T$ is a type $A \Rightarrow B \in \mathcal{S}_T$ with a term $\mathrm{app} \in \mathrm{Tm}B(\mathrm{sole}(A \Rightarrow B) \times \mathrm{sole}A)$ such that for any $M \in \mathrm{Tm}B(\Gamma \times \mathrm{sole}A)$ there exists a unique $\lambda M \in \mathrm{Tm}(A \Rightarrow B)\Gamma$ satisfying $\mathrm{app} * (\lambda M * \pi_1^{\Gamma,\mathrm{sole}A}, \pi_2^{\Gamma,\mathrm{sole}A}) = M$*
- *An empty type in $\mathcal{S}$ is a type $0 \in \mathcal{S}_T$ such that for every $\Gamma \in \mathcal{S}_c$, 0 is an initial object in Un_Γ.*
- *A sum type for $A_1, A_2 \in \mathcal{S}_T$ is a type $A_1 + A_2 \in \mathcal{S}_T$ with for each $\Gamma \in \mathcal{S}_C$ a coproduct structure $(A_1 + A_2, i_1^\Gamma, i_2^\Gamma)$ for A_1, A_2 such that for every $\gamma \in \mathcal{S}_c(\Delta, \Gamma)$,*

  $$\mathrm{Un}_\gamma(i_1^\Gamma) = i_1^\Delta$$

  and

  $$\mathrm{Un}_\gamma(i_2^\Gamma) = i_2^\Delta$$

Definition 6. *Let $\mathcal{S}$ be a CT structure*

- *A binary coproduct structure for $a_1, a_2 \in \mathcal{C}$ consists of*
  - *An object $a_1 + a_2 \in \mathcal{C}$*
  - *Morphisms $i_1 : \mathcal{C}(a_1, a_1 + a_2)$ and $i_2 : \mathcal{C}(a_2, a_1 + a_2)$*
  - *Such that for every $g_1 : \mathcal{C}(a_1, b)$ and $g_2 : \mathcal{C}(a_2, b)$ there exists a unique $[g_1, g_2] : \mathcal{C}(a_1 + a_2, b)$ satisfying $[g_1, g_2] \circ i_1 = g_1$ and $[g_1, g_2] \circ i_2 = g_2$.*

Problem 1 Sums and Distributive coproducts

Let $\mathcal{C}$ be a category with a terminal object and all binary products, i.e., all finite products. In class we discussed that (almost tautologically) $\mathcal{C}$ has

- all exponentials if and only if self$\mathcal{C}$ has all function types.

Your task is to prove the following non-trivial correspondences:

1. $\mathcal{C}$ has a *distributive* initial object if and only if self$\mathcal{C}$ has an empty type.
2. For any $a, b \in \mathcal{C}$, $\mathcal{C}$ has a *distributive* coproduct of a and b if and only if self$\mathcal{C}$ has a sum type of a and b.

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Definition 7. *A CT structure homomorphism $F : \mathcal{S} \to \mathcal{T}$ consists of*

- *A functor $F_c : \mathcal{S}_c \to \mathcal{T}_c$ of context categories such that*
  - *If $1 \in \mathcal{S}_c$ is the chosen terminal object of $\mathcal{S}_c$ then $F_c 1$ is terminal in $\mathcal{T}_c$.*
  - *For every Γ_1, Γ_2, $F_c(\Gamma_1 \times \Gamma_2), F_c(\pi_1^{\Gamma_1,\Gamma_2}), F_c(\pi_2^{\Gamma_1,\Gamma_2})$ is a product structure for $F_c\Gamma_1, F_c\Gamma_2$ in $\mathcal{T}_c$.*
- *A function $F_T : \mathcal{S}_T \to \mathcal{T}_T$ of types and for each $A \in \mathcal{S}_T$, a natural transformation $F_{\mathrm{Tm}} : \mathrm{Tm}(A) \to \mathrm{Tm}(F_T A) \circ F_c^{op}$ such that*
  - *For each $A \in \mathcal{S}_T$, $(F_T(\mathrm{sole}A), F_{\mathrm{Tm}}(\mathrm{var}^A))$ is a singleton context structure for $F_T A$.*

Definition 8. *Let $\mathcal{S}, \mathcal{T}$ be CT structures such that $\mathcal{S}$ has a unit type 1 and all product types $(A_1 \times A_2, \pi_1, \pi_2)$ and let $F : \mathcal{S} \to \mathcal{T}$ be a homomorphism of CT structures.*

1. *F preserves the unit type if $F_T 1$ is a unit type in $\mathcal{T}$*
2. *F preserves product types if for every product type structure $(A_1 \times A_2, \pi_1, \pi_2)$ for A_1, A_2, $(F_T(A_1 \times A_2), F_{\mathrm{Tm}}(\pi_1), F_{\mathrm{Tm}}(\pi_2))$ is a product structure for FA_1, FA_2.*

Definition 9. *A homomorphism of CT structures $F : \mathcal{S} \to \mathcal{T}$ is* faithful *if for each $\Gamma \in \mathcal{S}_c$ and $A \in \mathcal{S}_T$, the function $F_{tm}^{A,\Gamma} : \mathrm{Tm}_{\mathcal{S}}(A)(\Gamma) \to \mathrm{Tm}_{\mathcal{T}}(FA)(F\Gamma)$ is injective.*

For the remainder, fix a set of base types Σ_0

Definition 10. *Define $\mathcal{L}(\times, 1)$ to be the syntactic CT structure for STT generated from the base types in Σ_0 and the connectives $1, \times$.*

- *$\mathcal{L}(\times, 1)_T$ is the set of STT types generated from base types and $1, \times$*
- *$\mathcal{L}(\times, 1)_c$ is the category of STT contexts and substitutions using base types and $1, \times$*
- *$\mathrm{Tm}_{\mathcal{L}(\times,1)}$ is the predicator of terms using base types and $1, \times$.*

Similarly define $\mathcal{L}(\times, 1, \Rightarrow)$ to be the syntactic CT structure for STT generated from base types in Σ_0 and the connectives $1, \times, \Rightarrow$.

Theorem 1 (Weak Initiality of Syntactic CT Structures)**.** *Let $\mathcal{S}$ be a CT structure and $\iota : \Sigma_0 \to \mathcal{S}_T$ a function.*

- *If $\mathcal{S}$ has unit and product types, then we can construct a homomorphism of CT structures (soundness)*

  $$[\![\cdot]\!]^{\iota} : \mathcal{L}(\times, 1) \to \mathcal{S}$$

  that preserves unit and product types and base types in that for every $X \in \Sigma_0$, $[\![\cdot]\!]^{i} = i(X)$.

  Furthermore (completeness) $[\![\cdot]\!]^{\iota}$ is essentially unique*, in that if $F : \mathcal{L}(\times, 1) \to \mathcal{S}$ is a homomorphism preserving unit types, product types and $F(X) = i(X)$ for every $X \in \Sigma_0$, then there is a unique natural isomorphism $\alpha_c : \mathcal{S}_c^{\mathcal{L}(\times,1)_C}([\![\cdot]\!]^{\iota}, F)$.*

- *An analogous theorem holds for $\mathcal{L}(\times, 1, \Rightarrow)$: if $\mathcal{S}$ has unit, product and function types, we can construct a homomorphism of CT structures (soundness)*

$$(\!|\cdot|\!)^{\iota} : \mathcal{L}(\times, 1, \Rightarrow) \to \mathcal{S}$$

that preserves unit, product, function types and base types.

Furthermore (completeness) $(\!|\cdot|\!)^{\iota}$ is essentially unique, in that if $F : \mathcal{L}(\times, 1, \Rightarrow) \to \mathcal{S}$ is a homomorphism preserving unit types, product types, function types and base types then there is a unique natural isomorphism $\alpha_C : {\mathcal{S}_c}^{\mathcal{L}(\times,1,\Rightarrow)_c}((\!|\cdot|\!)^{\iota}, F)$.

Definition 11. *Define a CT structure homomorphism $i : \mathcal{L}(\times, 1) \to \mathcal{L}(\times, 1, \Rightarrow)$, the inclusion of the smaller type theory into the larger one:*

$$\begin{aligned} i_c(\Gamma) &= \Gamma \\ i_c(\gamma) &= \gamma \\ i_{ty}(A) &= A \\ i_{tm}([M]) &= [M] \quad ([M] \text{ means the equivalence class of } M \text{ in the equational theory.}) \end{aligned}$$

Observe that this is a CT structure homomorphism and additionally preserves product types and the unit type.

Problem 2 Conservativity of Adding Function Types to STT

Our goal is to prove that adding function types to STT with product types results in a *conservative extension* of the equational theory. That is, we want to show for any $\Gamma \in \mathcal{L}(\times, 1)_c$ and $A \in \mathcal{L}(\times, 1)_{ty}$, and $\Gamma \vdash M : A$ and $\Gamma \vdash M' : A$, if $\Gamma \vdash M = M' : A$ is provable in $STT(\times, 1, \Rightarrow)$, then in fact $\Gamma \vdash M = M' : A$ is already provable in $STT(\times, 1)$. Unraveling definitions, this says precisely that the homomorphism i is *faithful*.

We will prove this using a generalization of the method we used in problem set 1[1].

1. Show that if $F : \mathcal{S} \to \mathcal{T}$ and $G : \mathcal{T} \to \mathcal{U}$ are homomorphisms of CT structures and $G \circ F$ is faithful then F is faithful.

2. Show that if $F : \mathcal{S} \to \mathcal{T}$ and $F' : \mathcal{S} \to \mathcal{T}$ are homomorphisms of CT structures and $\alpha_c \in {\mathcal{T}_c}^{\mathcal{S}_c}(F, F')$ is a natural isomorphism and F is faithful then F' is faithful.

3. Show that for any category $\mathcal{C}$, the category of predicators $\mathscr{P}\mathcal{C}$ is cartesian closed (HINT: the cartesian closed structure is a direct generalization of the Heyting algebra structure you constructed in PS1). Therefore self($\mathscr{P}\mathcal{C}$) has unit, binary products and function types.

[1] again, there is a more complex proof that proves conservativity when we additionally have sum types

4. Define for every C-T structure $\mathcal{S}$, a homomorphism $Y : S \to \text{self}(\mathscr{P}\mathcal{S}_c)$ (Hint: use the Yoneda embedding) that
   - is faithful (Hint: use the Yoneda lemma)
   - preserves unit and product types
5. Define a homomorphism of CT structures $G : \mathcal{L}(\times, 1, \Rightarrow) \to \text{self}(\mathscr{P}\mathcal{L}(\times, 1)_c)$ and a natural isomorphism between $G \circ i$ and Y. (Hint: use the soundness part of weak initiality for $\mathcal{L}(\times, 1, \Rightarrow)$ and the completeness part of weak initiality for $\mathcal{L}(\times, 1)$).
6. Conclude that i is faithful.

In fact, this functor i satisfies an additional property: it is also *full*, meaning that $i_{tm}^{A,\Gamma}$ is not just injective but also *surjective*. That is, for any $\Gamma \in \mathcal{L}(\times, 1)_c$ and $A \in \mathcal{L}(\times, 1, \Rightarrow)_T$, if $\Gamma \vdash M : A$ is a term in $STT(\times, 1, \Rightarrow)$ then there exists a term $\Gamma \vdash M' : A$ in $STT(\times, 1)$ such that $\Gamma \vdash M = M' : A$ is provable. This can be proven using a more complex, but similar construction. See Crole chapter 4.10 for a variant of this argument.

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