Problem Set 2: Simple Type Theory

Released: January 23, 2023
Due: February 01, 2023, 11:59pm
Last modified: Jan 31, 2023, 12:00pm

Submit your solutions to this homework on Canvas in a group of 2 or 3. Your solutions must be submitted in pdf produced using LaTeX.

Corrections/Modifications: The following have been fixed from the initial released version:

1. The rule of substitution $()[\gamma] = ()$ has been corrected.
2. The notation $\gamma, M/x$ was previously incorrectly written as $\gamma[M/x]$.
3. Extra parentheses were added to help disambiguate the meaning of $(\gamma(x))[\delta]$: This means first apply γ to the variable x to get a term $\gamma(x)$ and then apply the substitution δ to that term.
4. The rule for weakening was reversed.
5. Corrected formulation of Problem 2.6.
6. Correct due date, matching Canvas and the course website.
7. Fix a typo in part 1 of problem 2.
8. Amend 2.1 to only require representative cases
9. Add some hints as to which parts of problem 2 are by induction.

Problem 1 Exponential Isomorphisms

We say that $x : A \vdash M : B$ and $y : B \vdash N : A$ form an *isomorphism* if $x : A \vdash N[M/y] = x$ and $y : B \vdash M[N/x] = y : B$. In this case we say A and B are isomorphic, written $A \cong B$.

Construct the following isomorphisms (with proof):

1. $A \Rightarrow B \Rightarrow C \cong (A \times B) \Rightarrow C$
2. $A \Rightarrow (B \times C) \cong (A \Rightarrow B) \times (A \Rightarrow C)$

3. $(A \Rightarrow 1) \cong 1$
4. $(A + B) \Rightarrow C \cong (A \Rightarrow C) \times (B \Rightarrow C)$
5. $0 \Rightarrow C \cong 1$

This gives an idea of why $A \Rightarrow B$ is in category theory sometimes called the *exponential* and written B^A.

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Problem 2 Admissibile Rules in STT

When all variables are known to be distinct, substitution $M[N/x]$ can simply be defined as the replacement of x with N everywhere in the term M. This definition is the STT version of the admissibility of the substitution principle of IPL:

$$\frac{\Gamma \vdash M : A \qquad \Gamma, x : A \vdash N : B}{\Gamma \vdash N[M/x]} \textsc{Subst}(*)$$

The admissible principle of contraction can also be viewed as a textual substitution in the term:

$$\frac{\Gamma, x : A, y : A, \Delta \vdash M : C}{\Gamma, x : A, \Delta \vdash M[x/y] : C} \textsc{Contraction}(*)$$

On the other hand, the use of variables to stand for assumptions means that exchange and weakening have no effect on the proof term:

$$\frac{\Gamma, y : B, x : A, \Delta \vdash M : C}{\Gamma, x : A, y : B, \Delta \vdash M : C} \textsc{Exchange}(*) \qquad \frac{\Gamma, \Delta \vdash M : C}{\Gamma, x : A, \Delta \vdash M : C} \textsc{Weakening}(*)$$

In this exercise you will show all of these principles are admissible and additionally prove some *equations* about substitution.

The simplest inductive proofs involve an auxiliary notion. We define a substitution from Δ to Γ to be a function γ that for each variable $x : A \in \Gamma$ produces a term $\Delta \vdash \gamma(x) : A$. We write $\gamma : \Delta \to \Gamma$ to mean a substitution from Δ to Γ. We can then define an admissible action of substitution:

$$\frac{\gamma : \Delta \to \Gamma \qquad \Gamma \vdash M : A}{\Delta \vdash M[\gamma] : A} \textsc{GenSubst}$$

Defined by induction on M:

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\begin{aligned}
x[\gamma] &= \gamma(x) \\
f(M_1, \ldots,)[\gamma] &= f(M_1[\gamma], \ldots) \\
(M, N)[\gamma] &= (M[\gamma], N[\gamma]) \\
(\pi_j M)[\gamma] &= \pi_j M[\gamma] \\
()[\gamma] &= () \\
(i_j M)[\gamma] &= i_j M[\gamma] \\
(\mathrm{case}_+ \, M\{i_1 x_1 \to N_1 | i_2 x_2 \to N_2\})[\gamma] &= (\mathrm{case}_+ \, M[\gamma]\{i_1 x_1 \to N_1[\gamma, x_1/x_1] | i_2 x_2 \to N_2[\gamma, x_2/x_2]\}) \\
(\mathrm{case}_0 \, M\{\})[\gamma] &= \mathrm{case}_0 \, M[\gamma]\{\} \\
(\lambda x.M)[\gamma] &= \lambda x.M[\gamma, x/x] \\
(M N)[\gamma] &= M[\gamma] \, N[\gamma]
\end{aligned}
$$

Where the notation $\gamma, M/x$ is the extension of the the function to map x to M:

$$
\begin{aligned}
(\gamma, M/x)(y) &= M && (\text{if } x = y) \\
(\gamma, M/x)(y) &= \gamma(y) && (\text{if } x \neq y)
\end{aligned}
$$

Define the *identity* substitution $\mathrm{id}_\Gamma : \Gamma \to \Gamma$ to map each variable in Γ to itself: $\mathrm{id}(x) = x$.

Given $\gamma : \Delta \to \Gamma$ and $\delta : \Xi \to \Delta$, define the *composition* $\gamma \circ \delta : \Xi \to \Gamma$ as $(\gamma \circ \delta)(x) = (\gamma(x))[\delta]$.

Below assume $\gamma : \Delta \to \Gamma$, $\delta : \Xi \to \Delta$, $\xi : \Xi' \to \Xi$ and $\Gamma \vdash M : A$.

1. Show (by induction on the derivation of $\Gamma \vdash M : A$) that $\Delta \vdash M[\gamma] : A$, i.e., that the GenSubst typing rule is admissible. You only need to show the following representative cases: $M = x$, $M = f(M_0, \ldots)$, $M = \lambda x.M'$ and $M = M' \, N$.

2. Show that one-place substitution, weakening, exchange and contraction are all instances of GenSubst.

3. Show (by induction on M) that $M[\mathrm{id}_\Gamma] = M$. Note that this and the following equalities are exact syntactic equalities, you will not need to use any $\beta\eta$ rules to prove it.

4. Show that $\gamma \circ \mathrm{id}_\Delta = \gamma$ and $\mathrm{id}_\Gamma \circ \gamma = \gamma$.

5. Show (by induction on M) that $M[\gamma \circ \delta] = (M[\gamma])[\delta]$

6. Show as a corollary that if $x_2 : A_2 \vdash N_1 : A_1$ and $x_3 : A_3 \vdash N_2 : A_2$ and $x_4 : A_4 \vdash N_3 : A_3$ then $(N_1[N_2/x_2])[N_3/x_3] = N_1[N_2[N_3/x_3]/x_2]$.

7. Show that $(\gamma \circ \delta) \circ \xi = \gamma \circ (\delta \circ \xi)$

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