Lecture 13: C-T Structures II

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1 Review of the C-T structure

Last time, we defined the C-T structure as a semantics or a judgemental structure[1] for STT. Recall that a C-T structure S consists of the following data:

1. S_T a set for types
2. S_C a cartesian category for contexts and substitutions
3. $\forall A \in S_T.\mathrm{Tm}_S(A)$ a predicator on S_C
4. $\forall A \in S_T.\mathrm{Tm}_S(A)$ is representable

For the last part, we have different ways to discuss representability. For a context $\mathrm{sole}\, A \in S_C$, we could either say $Y(\mathrm{sole}\, A) \cong \mathrm{Tm}_S(A)$, or use the Yoneda lemma and say given $\mathrm{var}^A \in \mathrm{Tm}_S(A)(\mathrm{sole}\, A)$, forall $M \in \mathrm{Tm}_S(A)(\mathrm{sole}\, A)$, there exists a unique substitution $M/\mathrm{var} : S_C(\Gamma, \mathrm{sole}\, A)$ such that $\mathrm{var}^A * M/\mathrm{var} = M$. The two ways are equivalent by Yoneda Lemma; we'll mainly use the second one, but we'll use the first one when convenient.

We've seen the Lindenbaum Algebra as an example of C-T structure. We'll give another example of it, which is the "typical" models.

1.1 The "typical" models

While the separation of types and contexts is natural when modeling syntax, it is less natural in mathematical models such as the category of sets. When we constructed the set-theoretic semantics, we interpreted both types and contexts as sets and substitutions and terms as functions.

In such "typical" models, we construct a C-T structure directly from the cartesian category alone where types and contexts are both interpreted as objects of a category. Let $\mathcal{C}$ be a cartesian category, we can define a C-T strurcture $\mathrm{self}\, \mathcal{C}$:

[1] A judgemental structure meaning it models the basic "judgments": types, terms, contexts and substitutions, but not yet an interpretation for particular connectives.

1. $(\mathrm{self}\,\mathcal{C})_C = \mathcal{C}$
2. $(\mathrm{self}\,\mathcal{C})_T = \mathcal{C}_0$
3. $\mathrm{Tm}_{\mathrm{self}\,\mathcal{C}}(a) = Y(a)$
   Given a type and b context, $\mathrm{Tm}(a)(b) = \mathcal{C}(b, a)$.
4. $\mathrm{sole}\,a = a$

Note that there is no separate notion between the substitution and the terms. Then the representability of $\mathrm{Tm}_{\mathrm{self}\,\mathcal{C}}(a)$ is trivial. The universal morphism in this case is

$$\mathrm{var} \in \mathrm{Tm}_{\mathrm{self}\,\mathcal{C}}(a)(a) = \mathcal{C}(a, a)$$

which is just the identity morphism.

We've defined the semantics of IPL in a preorder P, which is equivalent to defining $\mathscr{L}(\mathrm{IPL}) \to P$ which satisfies:

1. is a monotone function
2. preserves meets, Heyting implication, and joins

And now we are going to do the same for C-T structures, first without any connectives.

Definition 1. *Let S, T be C-T structures, a C-T structure homomorphism $F : S \to T$ consists of*

1. $F_T : S_T \to T_T$ *function of types*
2. $F_C : S_C \to T_C$ *functor on cartesian categories that preserves the finite products*[2]*:*
   - *If* $1 \in S_C$ *is the terminal object in* S_c *then* $F(1) \in T_C$ *terminal.*
   - *If* $\Gamma_1 \times \Gamma_2 \in S_c$ *with projections* $\pi_i : \Gamma_1 \times \Gamma_2 \to \Gamma_i$ *are the chosen product in* S_c *then* $F(\Gamma_1 \times \Gamma_2)$ *with projections* $F(\pi_i) : F(\Gamma_1 \times \Gamma_2) \to F(\Gamma_i)$ *is a cartesian product in* S_c.
3. $\forall M \in \mathrm{Tm}_S(A)(\Gamma) \mapsto F(M) \in \mathrm{Tm}_T(F(A))(F(\Gamma))$ *such that*

$$F_{\mathrm{Tm}}(M * \gamma) = F_{\mathrm{Tm}}(M) * F_C(\gamma)$$

   .

4. *Given* $\mathrm{var} \in \mathrm{Tm}_S(A)(\mathrm{sole}\,A)$ *as above, then* $F_{\mathrm{Tm}}(\mathrm{var}) \in \mathrm{Tm}_T(F_T A)(F_c(\mathrm{sole}\,A))$ *satisfies the same property in* T*, i.e., for any* $M \in \mathrm{Tm}_T(F_T A)(\Gamma)$ *there exists a unique* $M/F(\mathrm{var})$ *such that* $F_{\mathrm{Tm}}(\mathrm{var}) * M/F(\mathrm{var}) = M$.

[2] in class we discussed a more general definition but the following is simpler and equivalent

We can see that every notion in the syntax gets mapped to the corresponding notion in the semantics.

The third rule introduces a denotation of term M, essentially expressing the fact that $[\![M[\gamma]]\!] = [\![M]\!] \circ [\![\gamma]\!]$. We can also describe this operation as a natural transformation, where the condition can be described as naturality:

$$F^A_{\mathrm{Tm}} \in \mathrm{Set}^{op}(\mathrm{Tm}_S(A), \mathrm{Tm}_T(F(A)) \circ F_C)$$

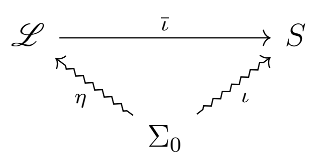
The fourth rule of preserving the variable corresponded to the fact that in our set-theoretic model, $x : A \vdash x : A$ was mapped to the identity function in Set.

Finally, we state without proof (as an exercise) that C-T structure homomorphisms form a category.

2 The Soundness Theorem of STT in C-T Structure

For starters, we can try to formulate the soundness theorem of STT$(\cdot)$ in C-T structure that only has empty set of types and only one context, which is the empty context.

For any C-T structure S, $\exists! \mathcal{L} \to S$ such that given the base types Σ_0, we have an interpretation ι from Σ_0 to S:



that is unique up to unique isomorphism.

2.1 Products of Types in a C-T structure

Then we want to define different soundness theorem for STT with whatever connectives picked in different C-T structures that contains some suitable notion of the object in STT. We also want to make sure that when we are defining the notions, we can connect them to C such that self C will have the appropriate structure.

Definition 2. *Let S be a C-T structure and $A, B \in S_T$,*

A product of A, B consists of

1. $P \in S_T$
2. $\mathrm{Tm}(P) \cong (\mathrm{Tm}(A)) \widetilde{\times} (\mathrm{Tm}(B))$ *where*

$$(\mathrm{Tm}(A) \widetilde{\times} \mathrm{Tm}(B))(\Gamma) := (\mathrm{Tm}(A)(\Gamma)) \times (\mathrm{Tm}(B)(\Gamma))$$

Reading this natural isomorphism intuitively it can be read as

$$\text{proving } \Gamma \vdash P \text{ is equivalent to proving } \Gamma \vdash A \text{ and } \Gamma \vdash B$$

.

Now, given this isomorphism of predicators, we can simplify this definition with the Yoneda Lemma. We first notice that $\text{Tm}(P)$ here is representable, meaning $\text{Tm}(P) \cong Y(\text{sole}\, P)$. By Yoneda Lemma, such an isomorphism is determined by a universal element

$$(\text{Tm}(A)\widetilde{\times}\text{Tm}(B))(\text{sole}\, P) := \text{Tm}(A)(\text{sole}\, P) \times \text{Tm}(B)(\text{sole}\, P)$$

where $\text{Tm}(A)(\text{sole}\, P)$ and $\text{Tm}(B)(\text{sole}\, P)$ are given exactly by π_1 and π_2:

$$x : A \times B \vdash \pi_1\, x : A \qquad\qquad x : A \times B \vdash \pi_2\, x : B$$

The universal property states that given any Γ, we can write down a natural transformation

$$(\text{Tm}(A)\widetilde{\times}\text{Tm}(B))(\Gamma) \longrightarrow \text{Tm}(P)(\Gamma)$$

, which is exactly the introduction rule:

$$\frac{\Gamma \vdash M_1 : A \qquad \Gamma \vdash M_2 : B}{\Gamma \vdash (M_1, M_2) : P}$$

The naturality here is that we are natural in the action of substitution under Γ which requires us to define substitution exactly as

$$(M_1, M_2) * \gamma = (M_1 * \gamma, M_2 * \gamma)$$

, which is builtin to the fact that $\text{Tm}(A)\widetilde{\times}\text{Tm}(B) \cong Y(\text{sole}\, P)$ is an isomorphism.

It's worthwhile noting that the two directions of the isomorphism correspond to the β and η rules for the STT.

For one direction:

$$\text{Tm}(A)\widetilde{\times}\text{Tm}(B) \xrightarrow{\times I} \text{Tm}(P) \xrightarrow{-/x} Y(\text{sole}\, P) \xrightarrow{(\pi_1 \circ -, \pi_2 \circ -)} \text{Tm}(A)\widetilde{\times}\text{Tm}(B)$$

$$\begin{array}{lll}
 & (\Gamma \vdash M_1 : A, \Gamma \vdash M_2 : B) & \\
\mapsto & \Gamma \vdash (M_1, M_2) : P & \\
\mapsto & (M_1, M_2)/x : \Gamma \to x : P & \\
\mapsto & (\pi_1\, x, \pi_2\, x) * (M_1, M_2)/x & \\
= & (\pi_1\, (M_1, M_2), \pi_2\, (M_1, M_2)) & \text{(subst.)} \\
= & (M_1, M_2) & (\beta\text{-rule})
\end{array}$$

where the last step defines the β-rule.

For the other:

$$\mathrm{Tm}(P) \xrightarrow{-/x} Y(\mathrm{sole}\, P) \xrightarrow{(\pi_1\circ -, \pi_2\circ -)} \mathrm{Tm}(A)\widetilde{\times}\mathrm{Tm}(B) \xrightarrow{\times I} \mathrm{Tm}(P)$$

$$\begin{aligned} & \Gamma \vdash M : P \\ \mapsto\quad & M/x : \Gamma \to x : P \\ \mapsto\quad & (\Gamma \vdash \pi_1\, M : A, \Gamma \vdash \pi_2\, M : B) \\ \mapsto\quad & \Gamma \vdash (\pi_1\, M, \pi_2\, M) : P \\ =\quad & \Gamma \vdash M : P \qquad\qquad (\eta\text{-rule}) \end{aligned}$$

where the last step defines the η-rule.

Finally, let's state that given $F : S \to T$ a C-T structure homomorphism, F preserves product types: $\forall A, B, P. \pi_{1,2} \in \mathrm{Tm}(A)(\mathrm{sole}\, P) \times \mathrm{Tm}(B)(\mathrm{sole}\, P)$ such that $\forall\exists!$, the image $F(\pi_{1,2}) \in \mathrm{Tm}(F(A))(F(\mathrm{sole}\, P)) \times \mathrm{Tm}(F(B))(F(\mathrm{sole}\, P))$ such that $\forall\exists!$.

2.2 Exponential in C-T structure

Definition 3. *Let S be a C-T structure and $A, B \in S_T$,*

A function type structure $A \Rightarrow B$ consists of

1. $E \in S_T$
2. $\mathrm{Tm}(E) \cong (\mathrm{Tm}A)\widetilde{\Rightarrow}(\mathrm{Tm}B)$

The syntax is given because from

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda(x : A).M : A \Rightarrow B}$$

we can extract the definition:

$$(\mathrm{Tm}(A)\widetilde{\Rightarrow}\mathrm{Tm}(B))(\Gamma) := \mathrm{Tm}(B)(\Gamma \times \mathrm{sole}\, A)$$

and define the rules for $*$:

$$\frac{M \in \mathrm{Tm}(B)(\Gamma \times \mathrm{sole}\, A) \qquad \gamma : \Delta \to \Gamma}{M * (\gamma \circ \pi_1, \pi_2) \in \mathrm{Tm}(B)(\Delta \times \mathrm{sole}\, A)}$$

The key observation here is that the substitution into Γ is the same thing as the substitution $\Delta \to \Gamma$ and a term $\mathrm{Tm}(A)(\Delta) \times \mathrm{sole}\, A$. And the $(\gamma \circ \pi_1, \pi_2)$ part is actually $(\gamma, x/x)$ where π_1 corresponds to the weakening rule that is used when we go from $(\lambda x.M)[\gamma]$ to $\lambda x.M[\gamma, x/x]$.

Similar to the product types, the isomorphism $\mathrm{sole}\, E \cong \mathrm{Tm}(E)$ will give us the elimination rule: $(\mathrm{Tm}(A)\widetilde{\Rightarrow}\mathrm{Tm}(B))(\mathrm{sole}\, E)$ is **just** $\mathrm{Tm}(B)(\mathrm{sole}\, E \times \mathrm{sole}\, A)$. It corresponds to

$$f : A \Rightarrow B, x : A \vdash f\ x : B$$

which is the rule for application with no substitution built into it.