Lecture 12: C-T Structure I

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1 Review of Yoneda's lemma

Last time, we defined the following predicators on a category $\mathcal{C}$[1]:

$$\begin{aligned} \tilde{1}(c) &:= 1 \\ (a \tilde{\times} b)(c) &:= \mathcal{C}(c, a) \times \mathcal{C}(c, b) \\ (a \tilde{\Rightarrow} b)(c) &:= \mathcal{C}(c \times a, b) \end{aligned}$$

And we claimed that the data of a terminal object, product and exponential corresponded precisely to those predicators being *representable*.

Definition 1. *A* representation *of a predicator P on $\mathcal{C}$ is a pair of an object $a \in \mathcal{C}_0$ and an isomorphism between Ya and P in $Set^{\mathcal{C}^{op}}$.*

We say a predicator P on $\mathcal{C}$ is representable *if there exists some representation of it.*

Then for each of the predicators above, representability would mean we have natural isomorphisms[2]

$$\begin{aligned} Y(1, c) = \mathcal{C}(1, c) &\cong 1 \\ Y(a \times b, c) = \mathcal{C}(c, a \times b) &\cong \mathcal{C}(c, a) \times \mathcal{C}(c, b) \\ Y(a \Rightarrow b, c) = \mathcal{C}(c, a \Rightarrow b) &\cong \mathcal{C}(c \times a \times b) \end{aligned}$$

We claim that a natural isomorphism $Y(a \times b) \cong a \tilde{\times} b$ is equivalent to a product structure on $a \times b$. Fix an isomorphism:

$$Y(a \times b) \underset{j}{\overset{i}{\rightleftarrows}} a \tilde{\times} b \ .$$

[1] the last one assumes the category has products
[2] i.e., an isomorphism in the functor category $\text{Set}^{\mathcal{C}^{op}}$

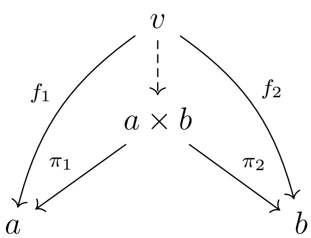
At first glance this doesn't look that much like a product structure, after all where did we construct any natural transformations in the definition of a product? Notice that i here has exactly the right domain for the Yoneda lemma to apply! By Yoneda lemma,

$$i \in \mathrm{Set}^{\mathcal{C}^{op}}(Y(a \times b), a \tilde{\times} b)$$

is uniquely determined by an element in

$$i(\mathrm{id}_{a \times b}) \in (a \tilde{\times} b)(a \times b) = \mathcal{C}(a \times b, a) \times \mathcal{C}(a \times b, b)$$

This should look familiar: these are the projections $\pi_1 \in \mathcal{C}(a \times b, a), \pi_2 \in \mathcal{C}(a \times b, b)$ in the definition of products before:



The proof of the Yoneda lemma showed that we could reconstruct i as follows:

$$i^v(f) = (\pi_1 \circ f, \pi_2 \circ f)$$

Then the inverse transformation $j : (a \tilde{\times} b)(v) \to \mathcal{C}(v, a \times b)$ gives us the existence part of the universal property, given f_1, f_2 we get $j^v(f_1, f_2) \in \mathcal{C}(v, a \times b)$. Then the equation $i \circ j = \mathrm{id}$ means for any $(f_1, f_2) \in (a \tilde{\times} b)(v)$

$$\begin{aligned} i(j(f_1, f_2)) &= (\pi_1 \circ (j(f_1, f_2)), \pi_2 \circ (j(f_1, f_2))) \\ &= (f_1, f_2) \end{aligned}$$

Which corresponds to the fact that the diagram above commutes.

Then the other direction $j \circ i = \mathrm{id}$ corresponds to the uniqueness condition in the product. Specifically that for any $f \in \mathcal{C}(v, a \times b)$ we have

$$\begin{aligned} j(i(f)) &= j(\pi_1 \circ f, \pi_2 \circ f) \\ &= f \end{aligned}$$

This shows that if we have an isomorphism $(i, j) : Y(a \times b) \cong (a \tilde{\times} b)$ then we can get out a product structure on $a \times b$, courtesy of the Yoneda lemma. We can also go the other way around. Given projections π_1, π_2 we can construct i as $i(f) = (f \circ \pi_1, f \circ \pi_2)$ and for each v we can construct j^v using the existence part of the definition of a product, then the commuting and uniqueness properties tell us that for each v, $i^v \circ j^v = \mathrm{id}$ and $j^v \circ i^v = \mathrm{id}$. However, we have to show that the family of all such j^v is a *natural* transformation. The following lemma, left as an exercise, shows that this follows:

Lemma 1. *Let α be a natural transformation from F to G, which are functors $\mathcal{C} \to \mathcal{D}$. Then α is an isomorphism iff for every $a \in \mathcal{C}_0$, $\alpha^c : \mathcal{D}(Fa, Ga)$ is an isomorphism.*

So we see that the pair of morphisms $(\pi_1, \pi_2) \in (a\tilde{\times}b)(a \times b)$ play a special role, in that they determine the whole natural isomorphism. We call these the *universal element* of the predicator $a\tilde{\times}b$.

Definition 2. *Given a representation $i : Y(a) \to P$ of a predicator P, the* universal element *of P is $i(id_a) \in P(a)$.*

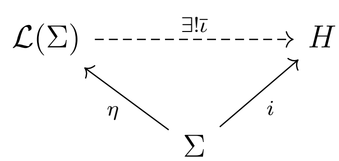
While the terminal object case is trivial, the exponential is interesting:

$$i(\mathrm{id}_{a\Rightarrow b}) \in (a\tilde{\Rightarrow}b)(a \Rightarrow b) = \mathcal{C}((a \Rightarrow b) \times a, b)$$

This is the application morphism we called app before.

2 Free Heyting Algebra Property of $IPL(\Sigma)$

Recall[3] that $IPL(\Sigma)$ presented the free Heyting algebra H from Σ in the following sense. First, we constructed a Heyting algebra $\mathcal{L}(\Sigma)$ from the syntax of IPL: the elements were equivalence classes . First, there is a *universal* interpretation $\eta : \Sigma \rightsquigarrow \mathcal{L}(\Sigma)$. Universality means for any other interpretation $i : \Sigma \rightsquigarrow H$, there exists a unique $\bar{\imath} : \mathcal{L}(\Sigma) \to H$ making the following diagram commute:



We can formalize this in terms of predicators as follows. First, η should be our universal element $\eta \in P(\mathcal{L}(\Sigma))$ for some *contravariant* predicator P on the category of Heyting algebras and homomorphisms. So $P(H)$ should have as elements *interpretations of* Σ, so let's call it Interp(Σ). So we have $\mathrm{Interp}(\Sigma)(H) = \Sigma \rightsquigarrow H$. This is a contravariant predicator where the composition of an interpretation $i : \Sigma \rightsquigarrow H$ and a homomorphism $f : H \to K$ is the composition $f \circ i$ of the underlying functions. Then the universal property of $IPL(\Sigma)$ is that the universal interpretation determines a natural isomorphism $Y^{op}\mathcal{L}(\Sigma) \cong \mathrm{Interp}(\Sigma)$, so we get for each H, a bijection of sets

$$\mathrm{HeytingAlg}(\mathcal{L}(\Sigma), H) \cong \Sigma \rightsquigarrow H$$

3 Introducing C-T structure

Now we want to define a *similar* universal property for the syntax of simple type theory that it is a free category with some structure in that it has a universal interpretation of the signature. However there are two reasons that things will be more complicated than before.

[3]note that this is a different formulation from the presentation we gave in class

First, recall that for Heyting algebra, Heyting implication, meets and joins in a poset are unique up to "=". But, we saw that exponentials, products and coproducts are unique, but we still have a *weaker* form of uniqueness - "unique up to unique isomorphism". This means that we will need to similarly weaken the uniqueness in the universal property for STT: the homomorphism we construct cannot be truly unique, only unique up to unique isomorphism.

Second, with IPL, things were simple enough that we could conflate the contexts Γ with the conjunction of all of the propositions $\bigwedge\Gamma$ without much trouble. However, the semantics of STT is strictly more complicated, so we will be more careful and use a notion of model where types and contexts are not conflated. In practice, the most common models will not require this distinction, but it is useful to maintain to make the soundess/completeness/freeness theorems easier to prove.

One benefit of this approach is that we will get highly modular soundness/completeness theorems for each choice of connective. We summarize them as follows:

Syntax	sound and complete notion of model	typical models
$\mathcal{STT}(\cdot)$	C-T structure	cartesian categories
$\mathcal{UTT}(\cdot)$	Category	category
$\mathcal{STT}(1, x)$	C-T structures with fin. products	cartesian categories
$\mathcal{STT}(\Rightarrow)$	C-T structures with exponentials	cartesian closed categories [4]
$\mathcal{STT}(\Rightarrow, x, 1)$	C-T structure with fin. products & exps	cartesian closed categories
$\mathcal{STT}(0, +)$	C-T structure with fin. coproducts	distributive bicartesian categories [5]
$\mathcal{STT}(0, +, \Rightarrow, \times, 1)$	C-T structure with all of the above	bicartesian closed category

Note that as with posets models of IPL, where joins were required to distribute over meets, as well here in the typical models the coproducts will need to be distributive over the products. However in the version with all of the connectives we don't need to additionally require that the coproducts are distributive because of the following lemmas:

Lemma 2. *If Ya and Yb are isomorphic in $Set^{\mathcal{C}^{op}}$ then so are a and b in $\mathcal{C}$.*

Proof. Let $i \in \mathrm{Set}^{\mathcal{C}^{op}}(Ya, Yb)$ be an isomorphism with inverse $i^{-1} \in \mathrm{Set}^{\mathcal{C}^{op}}(Yb, Ya)$. Then $i(\mathrm{id}_a) \in \mathcal{C}(a, b)$ and $i^{-1}(\mathrm{id}_b) \in \mathcal{C}(b, a)$ are an isomorphism. □

Lemma 3. *If a category $\mathcal{C}$ has finite products, finite coproducts and exponentials, then coproducts distributed over products in that:*

- $A \times 0 \cong 0$
- $A \times (B + C) \cong (A \times B) + (A \times C)$

[1]cartesian category with all exponentials.

[2]Distributive means that $A \times 0 \cong 0$ and $A \times (B + C) \cong A \times B + A \times C$.

Proof. By the prior lemma, it is sufficient to show the images under Y^{op} are isomorphic. But we can calculate using representability. For 0:

$$\begin{aligned} Y^{op}(D)(A \times 0) &\cong \mathcal{C}(A \times 0, D) \\ &\cong \mathcal{C}(0, A \Rightarrow D) \\ &\cong 1 \\ &\cong \mathcal{C}(0, D) \\ &\cong Y^{op}(D)(0) \end{aligned}$$

For +:

$$\begin{aligned} Y^{\mathrm{op}}(A \times (B + C))(D) &\cong \mathcal{C}(A \times (B + C), D) \\ &\cong \mathcal{C}(B + C, A \Rightarrow D) \\ &\cong \mathcal{C}(B, A \Rightarrow D) \times \mathcal{C}(C, A \Rightarrow D) \\ &\cong \mathcal{C}(A \times B, D) \times \mathcal{C}(A \times C, D) \\ &\cong \mathcal{C}(A \times B + A \times C, D) \\ &\cong Y^{\mathrm{op}}(A \times B + A \times C)(D). \end{aligned}$$

□

3.1 C-T structures

Next, we give the basic notion of a model of the judgments of STT, which we call a *C-T structure*, meaning a structure of a "category" with a notion of "type". We want this to abstract over the basic components of STT: types, terms, contexts and substitutions and the algebraic laws that they satisfy.

Definition 3. *A* ***C-T structure*** *S consists of*

1. *A set S_T we call the "types" and so write as $A, B, C, \ldots$*

2. *A cartesian category S_C. We think of the objects as contexts and write them $\Gamma, \Delta, \ldots$ and think of morphisms as substitutions and write them as $\gamma, \delta, \ldots$.*

3. *For each type A in S_T, we have a predicator $\mathrm{Tm}_S(A)$ on S_C.*

4. *For each $A \in S_T$, $\mathrm{Tm}_S(A)$ is representable: there exists $\mathrm{sole}(A)$ such that $Y(\mathrm{sole}(A)) \cong \mathrm{Tm}_S(A)$.*

Fix a choice of signature and connectives. We define a C-T structure $\mathcal{L}$ as follows.

types $\mathcal{L}_T$: types of STT.

contexts and substitutions The objects of $\mathcal{L}_c$ are the contexts of STT and the morphisms are the substitutions, i.e., a morphism $\gamma \in \mathcal{L}(\Delta, \Gamma)$ is a substitution $\gamma : \Delta \to \Gamma$.

We need to show that this category is *cartesian*:

- **Terminal object**: the terminal object is the *empty* context $\cdot$. This is because there is a unique substitution $\Delta \to \cdot$ for any Δ.
- **Cartesian products**: given contexts Γ_1 and Γ_2 whose variable names are disjoint, their cartesian product is given by the concatenation Γ_1, Γ_2, with the projections being substitutions

  $$\pi_1^{\Gamma_1,\Gamma_2} : \Gamma_1, \Gamma_2 \to \Gamma_1$$
  $$\pi_2^{\Gamma_1,\Gamma_2} : \Gamma_1, \Gamma_2 \to \Gamma_2$$

  defined syntactically as

  $$\pi_i(x) = x$$

  These correspond to the syntactic *weakening* of a term. Then to show the unique existence property, we have

  $$\frac{\gamma_1 : \Delta \to \Gamma_1 \quad \gamma_2 : \Delta \to \Gamma_2}{(\gamma_1, \gamma_2) : \Delta \to \Gamma_1, \Gamma_2}$$

  where the substitution (γ_1, γ_2) is defined in the following way:

  $$(\gamma_1, \gamma_2)(x : A \in \Gamma_1, \Gamma_2) = \begin{cases} \gamma_1(x) & x : A \in \Gamma_1 \\ \gamma_2(x) & x : A \in \Gamma_2. \end{cases}$$

  Then we can verify that $\pi_i^{\Gamma_1,\Gamma_2} \circ (\gamma_1, \gamma_2) = \gamma_i$ and it is the unique such substitution.

Terms For any $A \in \mathcal{L}_T$, we can define $\mathrm{Tm}_{\mathcal{L}}(A)(\Gamma)$ to be the set of terms $\{M | \Gamma \vdash M : A\}$. The operation $M * \gamma$ is defined to be substitution $M[\gamma]$, which we have shown previously to satisfy the predicator equations.

Then we need to show that for every $A \in \mathcal{L}_T$, $\mathrm{Tm}_{\mathcal{L}}(A)$ is representable. That is we need some context $\mathrm{Sole}(A)$ with $Y(\mathrm{Sole}(A)) \cong \mathrm{Tm}(\mathcal{L})(A)$, i.e., that there is a natural family of isomorphisms

$$\Gamma \to \mathrm{Sole}(A) \cong \mathrm{Tm}_{\mathcal{L}}(A)(\Gamma)$$

By the Yoneda lemma, this isomorphism is determined by a universal element:

$$\mathrm{var}^A \in \mathrm{Tm}_{\mathcal{L}}(A)(\mathrm{sole}(A))$$

We define $\mathrm{Sole}(A)$ to be a single variable context $x : A \in \mathrm{Ctx}$ for some arbitrary choice of variable name x, and define the universal element to be the variable:

$$\mathrm{var}^A = x : A \vdash x : A$$

Then universality means we need to show for all $M \in \mathrm{Tm}_{\mathcal{L}}(A)(\Gamma)$ there exists a unique substitution

$$M/x : \Gamma \to x : A$$

such that $x[M/x] = M$. As suggested by the notation we define $(M/x)(x) = M$. Then clearly $x[M/x] = M$ and it is the unique substitution from Γ to $x : A$ with this property.