Lecture 11: Universal Properties III: Representability and Yoneda's Lemma

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1 Predicator Examples

1.1 Definition

A predicator P on $\mathcal{C}$ consists of:

1. For each $a \in \mathcal{C}_0$, a set $P(a)$
2. For each $a, b \in \mathcal{C}_0$, a function $*^{a,b} : P(a) \times \mathcal{C}_1(b, a) \to P(b)$. That is:

$$\frac{\varphi \in P(a) \qquad f \in \mathcal{C}(b, a)}{\varphi * f \in P(b)}$$

3. Satisfying that $\varphi * \mathrm{id} = \varphi$ and $\varphi * (f \circ g) = (\varphi * f) * g$ (for appropriate domains and codomains).

For each of the following examples, the definition for $*$ will be given in the position of $\varphi * f$ in the above definition.

We can also define a predicator as just a functor from $\mathcal{C}^o \to \mathsf{Set}$.

This is a generalization over downward closed subsets. a preorder X is a subset $S \subseteq |X|$ such that whenever $x \in S$ and $y \leq x$, we have $y \in S$. A downward closed subset is just a monotone function $P^o \to 2$.

1.2 Terminal Predicator

Fix a category $\mathcal{C}$. Define a predicator $\tilde{1}$ on $\mathcal{C}$:

$$\tilde{1}(c) := \{\cdot\}$$

$$\frac{\cdot \in \tilde{1}(c) \qquad f \in \mathcal{C}(d, c)}{\cdot \in \tilde{1}(d)}$$

notice that $\cdot * f$ must be $\cdot$ since that is the only element in $\tilde{1}(d)$. Because of this, identity and associativity are trivially satisfied:

$$\begin{aligned} \varphi * \mathrm{id} &= \cdot \\ &= \varphi \end{aligned}$$

$$\begin{aligned} \varphi * (f \circ g) &= \cdot \\ &= \cdot * g \\ &= (\varphi * f) * g \end{aligned}$$

1.3 Product Predicator

Fix objects a, b in a category $\mathcal{C}$

$$(a \tilde{\times} b)(c) = \mathcal{C}(c, a) \times \mathcal{C}(c, b)$$

$$\frac{(g_1, g_2) \in (a \tilde{\times} b)(c) \qquad f \in \mathcal{C}(d, c)}{(g_1 \circ f, g_2 \circ f) \in \mathcal{C}(d, a) \times \mathcal{C}(d, b)}$$

This satisfies identity and associativity:

$$\begin{aligned} \varphi * \mathrm{id} &= (g_1, g_2) * \mathrm{id} \\ &= (g_1 \circ \mathrm{id}, g_2 \circ \mathrm{id}) \\ &= (g_1, g_2) \\ &= \varphi \end{aligned}$$

$$\begin{aligned} \varphi * (f \circ g) &= (h_1, h_2) * (f \circ g) \\ &= (h_1 \circ (f \circ g), h_2 \circ (f \circ g)) \\ &= ((h_1 \circ f) \circ g, (h_2 \circ f) \circ g) \\ &= (h_1 \circ f, h_2 \circ f) * g \\ &= ((h_1, h_2) * f) * g \\ &= (\varphi * f) * g \end{aligned}$$

1.4 exponential predicator

A Cartesian category is a category with all finite products. Fix a, b in a Cartesian category $\mathcal{C}$.

$$(a \tilde{\Rightarrow} b)(c) := \mathcal{C}(c \times a, b)$$

$$\frac{\varphi \in \mathcal{C}(c \times a, b) \qquad f \in \mathcal{C}(d, c)}{\varphi \circ (f \circ \pi_1, \pi_2) \in \mathcal{C}(d \times a, b)}$$

That is, $\varphi * f$ makes the following diagram commute, where the dotted arrow is the morphism guaranteed by the product $c \times a$:

$$
\begin{tikzcd}
c & & c \times a \arrow[ll, "\pi_1"'] \arrow[dl, "\pi_2"'] \arrow[dr, "\varphi"] & \\
& a & & b \\
d \arrow[uu, "f"] & & d \times a \arrow[ll, "\pi_1"] \arrow[ul, "\pi_2"] \arrow[ur, "\varphi * f"'] \arrow[uu, dashed, "(f \circ \pi_1, \pi_2)"] &
\end{tikzcd}
$$

This satisfies identity and associativity:

$$
\begin{aligned}
\varphi * \mathrm{id} &= \varphi \circ (\pi_1 \circ \mathrm{id}, \pi_2) \\
&= \varphi \circ (\pi_1, \pi_2) \\
&= \varphi
\end{aligned}
$$

$$
\begin{aligned}
\varphi * (f \circ g) &= \varphi \circ (f \circ g \circ \pi_1, \pi_2) \\
&= \varphi \circ (f \circ \pi_1, \pi_2) \circ (g \circ \pi_1, \pi_2) \\
&= (\varphi * f) \circ (g \circ \pi_1, \pi_2) \\
&= (\varphi * f) * g
\end{aligned}
$$

1.5 Initial and Coproduct Predicators

A copredicator is just a predicator on $\mathcal{C}^{op}$. We define the initial predicator as the terminal predicator in $\mathcal{C}^o$ and the coproduct predicator as the product predicator in $\mathcal{C}^o$

1.6 Free Monoid Predicator

Fix a set X. We will define a predicator Interp_X on Mon^{op}.

$$\mathrm{Interp}_X(M) := X \to |M|$$

$$\frac{i \in X \to |M| \qquad h \in \mathsf{Mon}_1(M, N)}{h \circ i \in X \to |N|}$$

Here, we are using the $\to$ operator to denote the set of functions between the two sets. We call this interp because it is directly analogous to the interpretations defined in PS3.

1.7 Representable Predicators

Fix an object $a \in \mathcal{C}$. Define the predicator $Y a$ on $\mathcal{C}$ as:

$$(Ya)(c) := \mathcal{C}(c, a)$$

$$\frac{\varphi \in (Ya)(c) \qquad f \in \mathcal{C}(d, c)}{\varphi \circ f \in (Ya)(d)}$$

In other words, our $*$ operation is exactly composition in the category, so identity and associativity are satisfied by definition of category.

2 Yoneda's Lemma

2.1 Yoneda Embedding

Y defined in the previous section is known as the Yoneda embedding. The Yoneda embedding tells us that for any object, we can define a predicator as the collection of morphisms into the object. This should inform how we think about all predicators as an abstract notion of morphisms into the object. This explains why the laws exist: if the predicators truly define sets of morphisms, then we would be able to compose them with real morphisms, and they would satsify the unit and associativity axioms just as a category does. In this way, we can think of a predicator as a generalized object, and $P(a)$ as the set of "morphisms" $a \to P$.

On the other hand, a predicator P is also an object of the category $\mathsf{Set}^{\mathcal{C}^o}$, and $Y(a) \in \mathsf{Set}^{\mathcal{C}^o}$ for each a, so we get a second notion of a morphism $a \to P$ by looking at the morphisms in $\mathsf{Set}^{\mathcal{C}^o}$, that is the *natural transformations*:

$$\mathsf{Set}^{\mathcal{C}^o}(Y(a), P)$$

So we are left with two different notions of what it means to construct a morphism from an object a to a predicator P. But in fact these two notions are equivalent, a statement formalized by *Yoneda's lemma* which says for any $a \in \mathcal{C}$ and $P \in \mathsf{Set}^{\mathcal{C}^o}$ a predicator on $\mathcal{C}$ we get a bijection:

$$P(a) \cong \mathsf{Set}^{\mathcal{C}^o}(Ya, P)$$

This lemma is *extremely* convenient: it reduces the work of defining a certain kind of natural transformation from defining a family of functions and proving naturality to simply providing an element $P(a)$. We'll see that exactly these kind of transformations arise in the formalization of universal properties.

2.2 Yoneda Lemma for Preorders

We'll start with proving a simplified version of the Yoneda lemma, the Yoneda lemma for preorders.

Lemma 1. *Suppose we have a preorder P with a downward closed subset S and an element $x \in |P|$. Then $x \in |S|$ iff for all $y \in P$, if $y \leq x$, then $y \in S$.*

Proof. Right to left: let $y = x$. We know $x \leq x$ by reflexivity, so $x \in S$.
Left to right: this is the definition of downward-closed. □

How is this a simplified version of the full Yoneda lemma? Recall that a downward closed subset is just a monotone function from $P^o \to 2$. We define $(Yx)(y) := y \leq x$. The right side of the lemma is equivalent to $Y(x) \leq S$ using the ordering on monotone functions defined in PS1. So this says $x \in S$ if and only if $Y(x) \leq S$.

2.3 Proof of Yoneda's Lemma

Lemma 2. *Suppose we have a category $\mathcal{C}$ a predicator P on $\mathcal{C}$ and an object $a \in \mathcal{C}_0$ Then we can construct a bijection*

$$P(a) \cong \mathsf{Set}^{\mathcal{C}^o}(Ya, P)$$

To aid with calculations, let's consider what it means concretely to have a natural transformation $\alpha \in \mathsf{Set}^{\mathcal{C}^o}(Ya, P)$. This says for any $c, d \in \mathcal{C}$ and $g : \mathcal{C}(d, c)$, we get a commuting diagram in the category of sets:

$$\begin{tikzcd}
(Ya)(c) = \mathcal{C}(c, a) \arrow[r, "\alpha^c"'] \arrow[d, "- \circ g"] & P(c) \arrow[d, "- * g"'] \\
(Ya)(d) = \mathcal{C}(d, a) \arrow[r, "\alpha^d"] & P(d)
\end{tikzcd}$$

That is, for any $f \in \mathcal{C}(c, a)$,

$$\alpha^c(f) * g = \alpha^d(f \circ g)$$

So we can think of α as a kind of "equivariant" map in the terminology of monoid/group actions.

Now we proceed by generalizing our previous proof from downward-closed subsets to predicators. We need to construct a bijection between $P(a)$ and $\mathsf{Set}^{\mathcal{C}^o}(Ya, P)$. We do this by defining:

$$i : P(a) \to \mathsf{Set}^{\mathcal{C}^o}(Ya, P)$$

$$j : \mathsf{Set}^{\mathcal{C}^o}(Ya, P) \to P(a)$$

and then showing i and j are inverses.

2.3.1 Definition of i

$$i(\varphi)^c(f) := \varphi * f$$

where $f \in (Ya)(c) = \mathcal{C}(c, a)$. Note that this is analogous to using the downward closure property in the Yoneda lemma for preorders: downward closure is generalized to the operation of composition $*$.

In other words, $i(\varphi)$ is a family of functions which, for each $c \in \mathcal{C}_0$ defines a function $i(\varphi)^c : (Ya)(c) \to P(c)$.

Since $i(\varphi)$ must be an element of $\mathsf{Set}^{\mathcal{C}^o}$, we now need to show that it is natural. That is, we want to show that for any $c, d \in \mathcal{C}_0$ and $g \in \mathcal{C}_1(d, c)$, the following diagram commutes:

$$\begin{tikzcd}
(Ya)(c) = \mathcal{C}(c,a) \arrow[r, "\varphi * -"'] \arrow[d, "- \circ g"] & P(c) \arrow[d, "- * g"'] \\
(Ya)(d) = \mathcal{C}(d,a) \arrow[r, "\varphi * -"] & P(d)
\end{tikzcd}$$

Which we can see (or just use our notion of equivariance) that this amounts to

$$(\varphi * f) * g = \varphi * (f \circ g)$$

These two are equal by the associativity law for the $*$ operation.

2.3.2 Definition of $j : \mathsf{Set}^{\mathcal{C}^o}(Ya, P) \to P(a)$

$$j(\alpha) := \alpha^a(\mathrm{id}_a)$$

The intuition is that α is a family of functions which, for each $c \in \mathcal{C}$ defines a function $(Ya)(c) \to P(c)$. We pick c to be a, so we get a function $(Ya)(a) = \mathcal{C}_1(a, a) \to P(a)$ and use the only morphism we know exists, the identity. Notice how this is similar to the proof of the Yoneda lemma for preorders: there we used reflexivity, and here we use its generalization to the identity morphism.

2.3.3 Proof of isomorphism

Now we must show that j, i form an isomorphism. We start with $j \circ i = \mathrm{id}_{P(a)}$:

$$\begin{aligned}
(j \circ i)(\varphi) &= j(i(\varphi)) \\
&= i(\varphi)(\mathrm{id}_a) \\
&= \varphi * \mathrm{id}_a \\
&= \varphi
\end{aligned}$$

The last step holds by the unit law for $*$.

Next, we want $i \circ j = \mathrm{id}_{\mathsf{Set}^{\mathcal{C}^o}(Ya,P)}$. That is, for any natural transformation $\alpha : Ya \to P$, we want to show $(i \circ j)(\alpha) = \alpha$. To show that two natural transformations are equal, we need to show that they are equal on every object $a \in \mathcal{C}$, that is, that

$\alpha^c = (i \circ j)(\alpha)^c : Y(a)(c) \to P(c)$. Then to show these are equal functions it is sufficient to show for every $f \in Y(a)(c)$ that $\alpha^c(f) = (i \circ j)(\alpha)^c(f)$. The right hand side is equal to

$$\begin{aligned}(i \circ j)(\alpha)^c(f) &= i(j(\alpha))^c(f)\\ &= i(\alpha^a(\mathrm{id}_a))^c(f)\\ &= \alpha^a(\mathrm{id}_a) * f\end{aligned}$$

Since α is natural, the following diagram must commute:

$$\begin{tikzcd}(Ya)(a) = \mathcal{C}(a,a) \arrow[r, "\alpha^a"] \arrow[d, "- \circ f"'] & P(a) \arrow[d, "- * f"] \\ (Ya)(c) = \mathcal{C}(c,a) \arrow[r, "\alpha^c"] & P(c)\end{tikzcd}$$

And the result follows by this naturality where we pick $\mathrm{id}_a \in \mathcal{C}(a,a)$. Explicitly using our equivariance formulation:

$$\begin{aligned}\alpha^a(\mathrm{id}_a) * f &= \alpha^c(\mathrm{id}_a \circ f)\\ &= \alpha^c(f)\end{aligned}$$

2.4 Representations

We want to show the predicators defined earlier are related to the universal properties discussed in previous lectures.

A **representation** of a predicator P on $\mathcal{C}$ is a pair of

- An object a of $\mathcal{C}$
- An isomorphism $Ya \cong P$ in $\mathsf{Set}^{\mathcal{C}^o}$.

Such an isomorphism in a functor category is often called a *natural isomorphism* since it is an isomorphism in a category whose morphisms are natural transformations.

This gives us a way to discuss the universal properties defined earlier:

- 1 is a terminal object iff $Y1 \cong \tilde{1}$
- a product p for a, b is equivalent to the isomorphism $Y(p) \cong a \tilde{\times} b$
- an exponential e for a, b is equivalent to the isomorphism $Y(e) \cong a \tilde{\Rightarrow} b$.
- 0 is an initial object iff $Y^o 0 \cong \tilde{1}$
- a coproduct s for a, b is equivalent to the isomorphism $Y^o(s) \cong a \tilde{\times} b$

This is analogous to the way we defined the operators on a preorder P in terms of downsets in the following way:

- t is a top element iff $\downarrow t = P$
- m is a meet of x, y is equivalent to $\downarrow m = \{z | z \leq x \wedge z \leq y\}$
- e is a Heyting implication of x, y is equivalent to $\downarrow e = \{z | z \wedge x \leq y\}$
- b is a bottom element iff $\uparrow b = P$
- j is a join of x, y is equivalent to $\uparrow j = \{z | x \leq z \wedge y \leq z\}$

The downsets/upsets have to be replaced by the covariant/contravariatn Yoneda embeddings and the equalities of downward-closed sets have to be replace by natural isomorphism.

3 Addendum: Yoneda lemma for sets

There is an even simpler version of the Yoneda lemma that applies to sets rather than posets.

Lemma 3. *Suppose we have a set X, a subset S and an element $x \in X$. Then $x \in S$ if and only if for every $y \in X$ if $y = x$ then $y \in S$.*

This is in fact an instance of the Yoneda lemma for preorders by thinking of X as a discrete poset where $x \leq y := (x = y)$, in which case any subset of S is trivially downward closed.