Problem Set 4

February 7, 2022

Homework is due the midnight before class on the 17th. If you want to volunteer to present this problem, e-mail me.

Both problems involve the free bi-cartesian category (BiCC) generated from an empty graph. We present this in full here for reference.

First, the types

$$1\ \text{type} \qquad \frac{A\ \text{type} \qquad B\ \text{type}}{A \times B\ \text{type}} \qquad 0\ \text{type} \qquad \frac{A\ \text{type} \qquad B\ \text{type}}{A + B\ \text{type}}$$

Next, the terms

$$x : A \vdash x : A \qquad x : A \vdash () : 1 \qquad \frac{x : A \vdash t_1 : B_1 \qquad x : A \vdash t_2 : B_2}{x : A \vdash (t_1, t_2) : B_1 \times B_2}$$

$$\frac{x : A \vdash t : B_1 \times B_2}{x : A \vdash \pi_1 t : B_1} \qquad \frac{x : A \vdash t : B_1 \times B_2}{x : A \vdash \pi_2 t : B_2} \qquad \frac{x : A \vdash t : 0}{x : A \vdash \text{match}_0 t : B}$$

$$\frac{x : A \vdash t : A_1 + A_2 \qquad x_1 : A_1 \vdash t_1 : B \qquad x_2 : A_2 \vdash t_2 : B}{x : A \vdash \text{match}_+ t\{\sigma_1 x_1.t_1\}\{\sigma_2 x_2.t_2\} : B} \qquad \frac{x : A \vdash t : B_1}{x : A \vdash \sigma_1 t : B_1 + B_2}$$

$$\frac{x : A \vdash t : B_2}{x : A \vdash \sigma_2 t : B_1 + B_2}$$

And the definition of substitution $t[s/x]$ by induction on t:

$$\begin{aligned}
x[s/x] &= s \\
()[s/x] &= () \\
(t_1, t_2)[s/x] &= (t_1[s/x], t_2[s/x]) \\
(\pi_i t)[s/x] &= \pi_i t[s/x] \\
(\text{match}_0 t)[s/x] &= \text{match}_0 t[s/x] \\
(\text{match}_+ t\{\sigma_1 x_1.u_1\}\{\sigma_2 x_2.u_2\})[s/x] &= \text{match}_+ t[s/x]\{\sigma_1 x_1.u_1\}\{\sigma_2 x_2.u_2\} \\
(\sigma_i t)[s/x] &= \sigma_i t[s/x]
\end{aligned}$$

And finally the equational theory. Note that we implicitly assume in any equality $s = t$ that both are well-typed with the same typing.

$$t = t \qquad \frac{s = t}{t = s} \qquad \frac{s = t \quad t = u}{s = u} \qquad \frac{t = t'}{\pi_i t = \pi_i t'} \qquad \frac{t_1 = t_1' \quad t_2 = t_2'}{(t_1, t_2) = (t_1', t_2')}$$

$$\frac{t = t'}{\sigma_i t = \sigma_i t'} \qquad \frac{t = t' \quad s_1 = s_1' \quad s_2 = s_2'}{\text{match}_+ t\{\sigma_1 x_1.s_1\}\{\sigma_2 x_2.s_2\} = \text{match}_+ t'\{\sigma_1 x_1.s_1'\}\{\sigma_2 x_2.s_2'\}}$$

$$\frac{x : A \vdash t : 1}{t = ()} \qquad \pi_i(t_1, t_2) = t_i \qquad \frac{x : A \vdash t : B_1 \times B_2}{t = (\pi_1 t, \pi_2 t)}$$

$$\frac{x : A \vdash s : 0 \quad y : 0 \vdash t : B}{t[s/y] = \text{match}_0 s} \qquad \text{match}_+ \sigma_i t\{\sigma_1 x_1.s_1\}\{\sigma_2 x_2.s_2\} = s_i[t/x_i]$$

$$\frac{x : A \vdash s : A_1 + A_2 \quad y : A_1 + A_2 \vdash t : B}{t[s/y] = \text{match}_+ s\{\sigma_1 x_1.t[\sigma_1 x_1/y]\}\{\sigma_2 x_2.t[\sigma_2 x_2/y]\}}$$

Feel free to use the following lemma in your proofs below.

Lemma 1. *If $x : A \vdash t : B$ and $x : A \vdash t' : B$ and $y : C \vdash u : A$ and $y : C \vdash u' : A$, then if $t = t'$ is derivable and $u = u'$ is derivable then*

$$t[u/x] = t'[u'/x]$$

is derivable.

In class we showed that this presents a category FreeBiCC that is the free bi-cartesian category on the empty graph. When we specialize this theorem to the *empty* graph, we get a simpler formulation of the UMP: the FreeBiCC on the empty graph is initial in the category of bi-cartesian categories and bi-cartesian functors, i.e., functors that preserve bi-cartesian structure.

Lemma 2 (UMP of Free BiCC)**.** *The syntax of the free bi-cartesian category defines a category FreeBiCC that is an initial object in the category of bi-cartesian categories and bi-cartesian functors.*

Proof. Let $\mathcal{C}$ be a bi-cartesian category. To construct a map from FreeBiCC to $\mathcal{C}$, by the UMP of FreeBiCC as the free bi-cartesian category generated from the empty graph, it is sufficient to define a graph homomorphism $i : \emptyset \to U(\mathcal{C})$. Then $\hat{\imath}$: FreeBiCC $\to \mathcal{C}$ is

1. a functor that preserves products, coproducts, initial and terminal object, along with pairing, projection, co-pairing, injection.

2. The unique functor satisfying $U\hat{\imath} \circ \eta = i$ where $\eta : \emptyset \to U(\text{FreeBiCC})$ is a fixed graph homomorphism.

But note that since the empty graph is the initial object in the category of graphs, this uniqueness condition is satisfied by any functor preserving bi-cartesian structure, and so $\hat{\imath}$ is the unique arrow from FreeBiCC to $\mathcal{C}$. □

Problem 1 Duality

Notice that every type in the free BiCC has a dual: products are dual to coproducts and terminal objects are dual to initial objects.

To demonstrate this duality, define an *op* translation from the free BiCC to its opposite as follows.

1. Define a translation A^{op} for each type.
2. Define a translation t^{op} on terms such that if

   $$x : A \vdash t : B$$

   then

   $$k : B^{op} \vdash t^{op} : A^{op}$$

3. Show that the translation is an involution: $(A^{op})^{op} = A$ and for any t, $(t^{op})^{op} = t$ in the equational theory. Are there any t such that $(t^{op})^{op}$ is not syntactically identical to t? If so, provide an example.

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Problem 2 Logical Relations

We can interpret the terms of the free Bi-cartesian category on an empty graph as a very simple programming language or formal logic.

As a logic, we interpet 0 as falsehood, + as disjunction, 1 as trivial truth and $\times$ as conjunction. To be a useful logic, we should verify that there is no proof of false. Of course, there are proofs such as

$$x : 0 \times (1 + 1) \vdash \pi_1 x : 0$$

but we would like that there is no proof of false from a trivial assumption, i.e., there is no proof

$$x : 1 \vdash t : 0$$

This property is called *consistency* of a logic.

To be a useful programming language[1], we should verify that we can *evaluate* programs down to a result. Since we added no facilities for effects or recursion, all programs should run to some kind of value in finite time. To be concrete, we should prove that any program of type $1 + 1$ evaluates to $\sigma_1()$ or $\sigma_2()$, which we can think of as a boolean. The specification for our evaluator is that it should respect our notion of $\beta\eta$ equality, so t should evaluate to $\sigma_1()$ if and only if $t = \sigma_1()$ in our equational theory. Of course, there are terms of type $1 + 1$ that are not equal to $\sigma_1()$ or $\sigma_2()$, such as

$$x : 1 + 1 \vdash x : 1 + 1$$

[1] or *constructive* logic

instead we should restrict to programs that only take a trivial input. So we want to show that terms of the form:

$$x : 1 \vdash t : 1 + 1$$

Are all equal to $\sigma_1()$ or $\sigma_2()$. A programming languages that satisfies this or a similar property is called *normalizing*.

We should also make sure that our programming language specification is not trivial. In particular, since we did not add facilities for non-determinism, we should verify that if a program evaluates to $\sigma_1()$, then it cannot also evaluate to $\sigma_2()$. That is, we should show that $\sigma_1() \neq \sigma_2()$.

- Show that the terms $x : 1 \vdash \sigma_1() : 1 + 1$ and $x : 1 \vdash \sigma_2() : 1 + 1$ in the free BiCC on the empty graph are not equal. Hint: use the UMP of the free BiCC to construct a functor F from the free BiCC to another category where $F(\sigma_1()) \neq F(\sigma_2())$.

Showing consistency/normalization for a calculus can be difficult. One of the most robust proof techniques for proving this is called the method of *logical relations*. Let's use the method of logical relations to prove consistency and normalization of the free BiCC.

The method of logical relations in this case is centered on a category we call *LR* defined as folows:

1. The objects of LR are pairs (A, P) of a type A in the free BiCC and a subset of the terms of type A with trivial input: $P \subseteq \{t \,|\, x : 1 \vdash t : A\}$
2. A morphism from (A, P) to (B, Q) is a term $y : A \vdash s : B$ such that for any $t \in P$, $s[t/y] \in Q$.

We will then show that LR is itself a bi-cartesian category with the following definitions of the objects involved:

1. The initial object is $(0, \emptyset)$
2. The coproduct of (A, P) and (B, Q) is $(A + B, R_+)$ where R_+ is
$$\{u \,|\, (\exists t \in P.\ u = \sigma_1 t) \vee (\exists t \in Q.\ u = \sigma_2 t)\}$$
3. The terminal object is $(1, \top)$ where $\top = \{t \,|\, x : 1 \vdash t : 1\}$.
4. The product of (A, P) and (B, Q) is $(A \times B, R_\times)$ where $R_\times$ is
$$\{u \,|\, (x : 1 \vdash u : A \times B) \wedge \pi_1 u \in P \wedge \pi_2 u \in Q\}$$

Now, prove consistency and normalization for BiCC as follows:

1. Define identity and composition in LR, and prove it forms a category.

2. Show that the definitions above give LR the structure of a bi-cartesian category, and that the "forgetful" functor $U : \text{LR} \to \text{FreeBiCC}$ preserves the bi-cartesian structure, i.e., it takes products to products, projections to projections, etc.

3. Conclude using the UMP of the free bi-cartesian category that there is a functor from $s : \text{FreeBiCC} \to \text{LR}$ that preserves bi-cartesian structure and is a section of U, that is:

$$U \circ s = \text{id}_{FreeBiCC}$$

4. Prove as a corollary that

   (a) BiCC is consistent: there is no term $x : 1 \vdash t : 0$

   (b) BiCC is normalizing: any term $x : 1 \vdash t : 1 + 1$ is equal to $\sigma_1()$ or $\sigma_2()$ in the equational theory.

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