Lecture 22: Adjunctions & Adjoint Functors

Lecturer: Max S. New
Scribe: Christopher Davis

November 12th, 2025

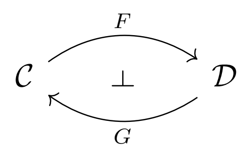
1 Adjunctions

Definition 1. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An* ***adjunction*** *between $\mathcal{C}$ and $\mathcal{D}$ is:*

- *A pair of functors: $F : \mathcal{C} \to \mathcal{D}$ (left adjoint) and $G : \mathcal{D} \to \mathcal{C}$ (right adjoint)*
- *A natural isomorphism for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$:*

$$\hom_{\mathcal{D}}(Fc, d) \cong \hom_{\mathcal{C}}(c, Gd)$$

We write this as $F \dashv G$.



The diagram above represents an adjunction $F \dashv G$, where:

- F is the left adjoint functor from $\mathcal{C}$ to $\mathcal{D}$
- G is the right adjoint functor from $\mathcal{D}$ to $\mathcal{C}$
- The symbol $\dashv$ indicates the adjunction relationship

We say F is left adjoint to G, or G is right adjoint to F.

This adjunction means there is a natural bijection: $\mathcal{D}(Fc, d) \cong \mathcal{C}(c, Gd)$ for all objects c in $\mathcal{C}$ and d in $\mathcal{D}$. I.e., a morphism out of the left adjoint in one category is equivalent to a morphism going into the right adjoint in the other category.

Definition 2 (Profunctor)**.** *Let $\mathcal{C}, \mathcal{D}$ be categories. A* ***profunctor*** *$p : \mathcal{C} \nrightarrow \mathcal{D}$ is a functor $p : \mathcal{D}^{op} \times \mathcal{C} \to Set$*

Profunctors can be considered generalizations of relations.

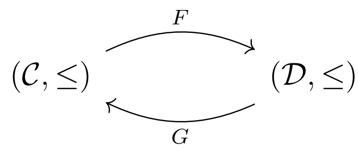
Remark 1. *An adjunction between $\mathcal{C}$ and $\mathcal{D}$ is a natural isomorphism between two profunctors $\mathcal{D} \nrightarrow \mathcal{C}$.*

Consider $F \dashv G$. In $\mathcal{D}(Fc, d) \cong \mathcal{C}(c, Gd)$, note c is in the contravariant position on both sides, and d the covariant. So, both sides stand for functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$. These are precisely profunctors $\mathcal{D} \nrightarrow \mathcal{C}$.

1.1 Examples

Definition 3 (Galois connection)**.**

Suppose $\mathcal{C}$ and $\mathcal{D}$ are posets (or pre-orders). Let F, G be monotone functions s.t.



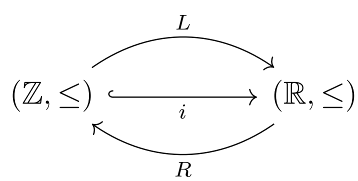
for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$, $Fc \leq_{\mathcal{D}} d$ if and only if $c \leq_{\mathcal{C}} Gd$.

This is called (for posets) a **Galois connection**.

Let's consider examples of Galois connections.

1.1.1 Galois Connection: $\mathbb{Z}$ and $\mathbb{R}$

Consider the usual orderings of $(\mathbb{Z}, \leq)$ and $(\mathbb{R}, \leq)$. Take the inclusion map $i : \mathbb{Z} \to \mathbb{R}$. This is clearly monotone and a suborder. Is this an adjunction?



What would the right adjoint be? Well, the condition here would be: $\forall z \in \mathbb{Z}$ and $r \in \mathbb{R}$, $i\ z \leq r$ iff. $z \leq R\ r \in \mathbb{Z}$. I.e., $i\ z = z$ is less than or equal to some integer that's a function of r. Then, what is R? Naturally, the ceiling or floor comes to mind. Note we have $1 \leq 1/2$ iff. $1 \leq R\ 1/2$, so $R \neq \lceil \cdot \rceil$. However, $R = \lfloor \cdot \rfloor$ works; i.e., $z \leq r \Leftrightarrow z \leq \lfloor r \rfloor$.

What about the left adjoint L? We want $L\ r \leq z$ iff. $r \leq i\ z$. Well, note that $\lceil r \rceil \leq z \Leftrightarrow r \leq z$. So, we can say $L = \lceil \cdot \rceil$.

This then defines an adjunction $\lceil \cdot \rceil \dashv \lfloor \cdot \rfloor$

1.1.2 Galois Connection: Propositions

Let's consider propositions with a provability ordering $(\text{Prop}, \vdash)$ and, on the other hand, families of propositions $\left(\text{Prop}^X, \vdash\right)$, where X is a set and provability here is pointwise. We can consider Prop^X as as propositions with a variable x, i.e.

$$\varphi(x) \vdash \psi(x)\ \forall x \in X \tag{1}$$

These propositions could simply be booleans, or it could be in a formal system of logic.

Similar to example 1.1.1, there is a form of inclusion here, $(\text{Prop}, \vdash) \hookrightarrow (\text{Prop}^X, \vdash)$, which we can think of as "weakening" the proposition. I.e., for the proposition φ in (1), we can weaken it with respect to the variable x to think of it as index family propositions. Or, we take each proposition to the constant function that returns that. We call this inclusion Δ.

$$(\text{Prop}, \vdash) \overset{\Delta}{\hookrightarrow} (\text{Prop}^X, \vdash) \qquad \text{where} \quad \Delta(\varphi)(x) = \varphi \tag{2}$$

Note that this is a monotone function. If $\varphi \vdash \psi$, then $\Delta(\varphi)(x) \vdash \Delta(\psi)(x)$.

Do we have a right and left adjoint here? A right adjoint means that, $\forall x$, $\Delta(\varphi)(x) \vdash \Delta(\psi)(x)$ iff. $\varphi \vdash R(\psi)$. So, what is this proposition $R(\psi)$? It is a universal quantifier. It's saying that we can prove $\forall x \,.\, \psi(x) \dashv \varphi$ iff. $\Delta(\varphi)(x) \vdash \Delta(\psi)(x), \forall x$. This is actually the rule of provability in first order logic; for adding free variable x,

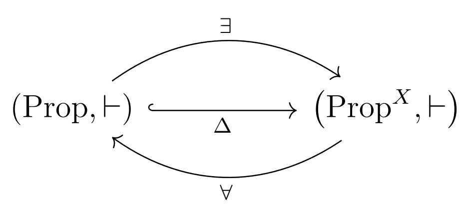
$$\frac{\varphi \vdash^x \psi(x)}{\overline{\varphi \vdash \forall x. \psi(x)}}$$

I.e. the for-all introduction principal.

Then, the left adjoint would mean $L(\psi) \vdash \varphi$ iff. $\forall x,\ \psi(x) \vdash \varphi = \Delta(\varphi)(x)$. This, in turn, will be the existential quantifier. The idea is that we can prove something follows from an existentially-quanitified statement if under any possible witness we could prove the statement. Symbolically, $\exists\, x \,.\, \psi(X) \vdash \varphi$, meaning we need to prove $\psi(x) \vdash \varphi$ for a free variable x (i.e., $\psi(x) \vdash^x \varphi$). In a sequence calculus presentation of first order logic, the existential quantifier is just the rule

$$\frac{\psi(x) \vdash^x \varphi}{\exists x. \psi(x) \vdash \varphi}$$

So, we're left with:



There is still some ambiguity here. For formalizing the adjunction, there are two setups. The first is to just take propositions as booleans and Prop^X as functions $X \to \text{Prop}$. Then, left/right adjoint will simply be existential/universal quantifiers being used to compute a boolean (or a proposition). On the other hand, we can take a fully-syntactic view and treat Prop^X as syntactic propositions. That is, not as functions $X \to \text{Prop}$, but rather as propositions with a free variable of type X.