Lecture 12: Initiality of STLC

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Things we need to remember about presheafs:

1. A presheaf is a functor $P : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$
2. A presheaf *is* a universal property
3. An object $x \in \mathcal{C}$ "has" the universal property P if

$$\mathrm{Yoneda} x \cong P$$

4. Equivalently, given a universal element $\eta : P(x)$, then $\eta \circ_p - : \mathrm{Yoneda} x \to P$ is a natural isomorphism

1 Weak Initiality of IPL(Σ)

1. IPL(Σ) form a biHeyting (pre)algebra, "tautological interpretation" is self of Σ in IPL(Σ).
2. For all interpretations σ in a biHeyting prealgebra P,
   (a) $[\![\cdot]\!] : \mathrm{IPL}(\Sigma) \to P$
      i. monotone
      ii. preserves biHeyting algebra structure; sends meets to meets, joins to joins, etc.
      iii. preserves the interpretation
   (b) We have a unique such function (up to $\leq, \geq$)

Syntax	**Sound & Complete Model**	**Typical Model**
$\mathrm{STLC}(0, +, 1, \times, \implies)$	$\mathrm{SCwF}(0, +, 1, \times, \implies)$	BiCartesian Closed Category
$\mathrm{STLC}(\emptyset)$	SCwF	Cartesian Categories
$\mathrm{STLC}(1, \times, \implies)$	$\mathrm{SCwF}(1, \times, \implies)$	Cartesian Closed Category
$\mathrm{STLC}(0, +, 1, \times)$	$\mathrm{SCwF}(0, +, 1, \times)$	Distributive BiCartesian Category
$\mathrm{STLC}(1, \times)$	$\mathrm{SCwF}(1, \times)$	Cartesian Categories

Definition 1 (biCartesian closed cateogry). *A biCartesian closed cateogry is a category $\mathcal{C}$ with $\forall A, B \in \mathcal{C}$ we have*

- *a coproduct, $A + B$, along with inclusion maps σ_1, σ_2*
- *a product $A \times B$, along with projection maps π_1, π_2*
- *exponentials $A \implies B$, along with an evaluation map,* $\mathrm{eval} : B^A \times A \to B$

Initial and terminal objects 1, 0.

- *Cartesian: 1, $A \times B$*
- *coCartesian: 0, $A + B$*
- *Cartesian closed: 1, $A \times B, A \implies B$*

Definition 2 (Distributive coproducts, initial objects). *Let $\mathcal{C}$ be a Cartesian category. A coproduct $A \times B$ in $\mathcal{C}$ is distributive when $\forall C$, we have*

$$(A + B) \times C \xleftarrow{\sim} (A \times C) + (B \times C)$$

as an isomorphism. An initial object is given when

$$0 \times C \xleftarrow{\sim} 0$$

is an isomorphism.

Coproduct is a "left-handed" universal property, meaning it's easy to map *out* of it. We can simply handle cases.

Theorem 1. *Every biCartesian closed category is distributive.*

Proof. First, let's show that $0 \times C \xleftarrow{\sim} 0$ is an isomorphism. The Yoneda embedding is fully-faithful, so, it suffices to show that $\mathrm{Yoneda}^{\mathrm{op}}(0 \times C) \xleftarrow{\sim} \mathrm{Yoneda}^{\mathrm{op}} 0$. For all $X \in \mathcal{C}$, we have

$$\begin{aligned} \mathcal{C}(0 \times C, X) &\cong \mathcal{C}(0, X^C) \\ &\cong 1 \\ &\cong \mathcal{C}(0, X) \end{aligned}$$

Now, to show that $(A + B) \times C \xleftarrow{\sim} (A \times C) + (B \times C)$ an isomorphism, we have

$$\begin{aligned} \mathcal{C}((A + B) \times C, X) &\cong \mathcal{C}(A + B, X^C) \\ &\cong \mathcal{C}(A, X^C) \times \mathcal{C}(B, X^C) \\ &\cong \mathcal{C}(A \times C, X) \times \mathcal{C}(B \times C, X) \\ &\cong \mathcal{C}((A \times C) + (B \times C), X) \end{aligned}$$

□

Definition 3 (Simple Category with Families)**.** *A simple category with families (SCwF), S consists of*

1. *A set S_t of "types"*
2. *A category S_C of "contexts and substitutions"*
3. *For every $A \in S_t$, we have a "presheaf of terms", $\mathrm{Tm}(A)$ on S_C*
4. *S_C has a terminal object $\bullet \in S_C$*
5. *$\forall \Gamma \in S_C, A \in S_t$, we have a "product context", $\Gamma \times A$ such that $Yoneda(\Gamma \times A) \cong Yoneda\ \Gamma \times Tm(A)$. I.e., $S_C(\Delta, \Gamma \times A) \cong S_C(\Delta, \Gamma) \times \mathrm{Tm}(A)\Delta$.*

Let us define a SCwF called $\mathrm{STLC}(\Sigma)(\ldots(\text{connectives}))$

1. Types are types of STLC
2. Contexts are (syntactic) contexts Γ. As in PS2, for $\gamma : \Delta \to \Gamma$, and $\forall (x : A) \in \Gamma$, we have $\Delta \vdash \gamma(x) : A$.
3. $\mathrm{Tm}(A)(\Gamma) := \{M \mid \Gamma \vdash M : A\}$, for $M \in \mathrm{Tm}(A)(\Gamma)$ and $\gamma : \Delta \to \Gamma$, we have presheaf action $M \circ \gamma := M[\gamma]$
4. We have the terminal context $\Gamma \to \bullet \cong 1$.
5. Let Γ, A. We want to construct a context, $\Gamma \times A$, such that $(\Delta \to \Gamma \times A) \cong (\Delta \to \Gamma) \times (\Delta \vdash \bullet : A)$. We can define $\Gamma \times A := \Gamma, x \in A$ be extending Γ with a free variable of the type A. Given $\gamma : \Delta \to \Gamma$ and $\Delta \vdash M : A$, we have $\gamma, M/x : \Delta \to \Gamma, x \in A$.

Let $\mathcal{C}$ be a Cartesian category. We will define a SCwF called $\mathrm{dem}(\mathcal{C})$.

1. Types are objects of $\mathcal{C}$
2. $\mathrm{dem}(\mathcal{C})_C := \mathcal{C}$
3. $\forall A \in C_0$ we have $\mathrm{dem}(\mathrm{Tm}(A)) = \mathrm{Yoneda}(A) : Psh(\mathcal{C})$
4. Terminal object, free
5. $\Gamma \times A$, free

Fix a SCwF S.

1. For $A, B \in S_t$, we define a product type $A \times B \in S_t$ with $\mathrm{Tm}_S(A \times B) \cong \mathrm{Tm}_S A \times \mathrm{Tm}_S B$. I.e., $\forall \Gamma$, we have $\mathrm{Tm}_S(A \times B)(\Gamma) \cong (\mathrm{Tm}_S A)(\Gamma) \times (\mathrm{Tm}_S B)(\Gamma)$

Theorem 2. $\mathrm{dem}(\mathcal{C})$ *always has product types $A \times B$, which are simply given by their product in $\mathcal{C}$.*

Theorem 3. *STLC($\cdots$, $\times$) has products*

$$\Gamma \vdash \bullet : A \times B \cong (\Gamma \vdash \bullet : A) \times (\Gamma \vdash \bullet : B)$$

Right to left corresponds to the product introduction rule, i.e., $(M, N)[\gamma] = (M[\gamma], N[\gamma])$*. Left to right corresponds to the product elimination rules, i.e.,* $(\pi_i M)[\gamma] = \pi_i(M[\gamma])$*.*

A unit type in S *is a type* $1 \in S_t$*. We have*

$$\mathrm{Tm}1 \cong 1$$
$$(\mathrm{Tm}1)(\Gamma) \cong \{+\}$$

A function type $A \Rightarrow B \in S$*, we have* $\mathrm{Tm}(A \Rightarrow B)(\Gamma) \cong \mathrm{Tm}(B)(\Gamma \times A)$

Theorem 4. *If STLC has function types in the syntactic sense then SCwF STLC has function types in the semantic sense. I.e.,* $(Mx)[\gamma, x/x] = M[\gamma]x$

In a SCwF S with types A, C, we have a "continuation presheaf", $\mathrm{Cont}AB$ on S_c with $(\mathrm{Cont}AB)(\Gamma) := (\mathrm{Tm}B)(\Gamma \times A)$. We have an "empty type"' 0 in a SCwF is a type $0 \in S_t$ with $(\mathrm{Cont}0C) \cong 1$ for all $C \in \mathcal{S}_{\sqcup}$. For $A, B \in S_t$, we have a "sum type"' $A + B \in S_t$ such that for all $C \in S_t$, we have $\mathrm{Cont}(A+B)C \cong \mathrm{Cont}AC \times \mathrm{Cont}BC$

For $\mathrm{dem}(\mathcal{C})$, having an empty type is equivalent to having a distributive initial object. For all $C \in S_t$, we have $\mathrm{Cont}0C \cong 1$, so for all Γ, C we have $\mathcal{C}(\Gamma \times 0, C) \cong 1 \cong \mathcal{C}(0, C)$, so $\Gamma \times 0 \cong 0$. Similarly, sum types correspond to a distributive coproduct.